UNRELIABLE SERVER, BULK QUEUE WITH TWO TYPES OF SERVICES, STATE DEPENDENT ARRIVAL RATES AND WITH COMPULSORY TYPE SERVICE VACATION

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Abstract

A bulk queue has been analyzed in this article. The arrival follows compound Poisson process with state dependent arrival rates. The system contains a single server with infinite waiting line. The server provides service to batch of customers with batch size is a fixed number K. Each batch has given two types of services called essential service (ES) and optional service (OS). The optional service is based on a Bernoulli distribution. The service times are generally distributed. Busy server may breakdown with the number of breakdowns follows a Poisson process. Immediately, the server under goes repair, the repair period is generally distributed. After completion of each service, the server takes compulsory vacation, the vacation period is generally distributed. This model is analyzed using supplementary variable technique. Some performance measures are derived. Particular and numerical models are also presented, to show the reliability of the model discussed in this paper.

Keywords: Non-Markovian queue - Bulk arrival queue - Compulsory vacation - Essential and Optional service - Unreliable server - State dependent rates.
2000 Mathematics Subject Classification Number: 90B22,60K25,60K30.

1. Introduction

After serving a customer or a batch of customers, if there are no customers in the system to attend, the server takes a predetermined break. The period of breaks is called vacation. The duration and frequency of these vacations can vary based on the system's design. Various modifications have been made by researchers on the vacation period one such modification is compulsory vacation. A compulsory vacation refers to a system where the server is mandated to take vacations, after completing certain tasks or under specific conditions. These vacations are integral to the system's operation and are not optional. Compulsory vacation queues are useful in modelling systems where servers require regular vacations, such as machinery needing maintenance or employees taking scheduled rest periods. They help in understanding the impact of these vacations on system performance, including customer wait times and queue lengths. Understanding compulsory vacation queues is essential for designing efficient systems that balance service quality with necessary downtime, ensuring optimal performance and customer satisfaction. The earlier works on vacation systems are by Cooper (1970,1981), Levy and Yechiali (1975), Keilson and servi (1986), and Doshi (1990). Madan (1992) analyzed single server queue with compulsory server vacation. Kalyanaraman and Suvitha (2012) have considered an M/G/1 queue with compulsory server vacation and with two types services, restricted admissibility. Vanitha (2015) has analyzed a compulsory vacation queue with batch service. Kalyanaraman and Nagarajan (2016) considered a bulk arrival, fixed batch service queue with unreliable server and with compulsory server vacation.

The other topic is state-dependent input and output mechanisms in queueing systems, usually called the state-dependent controls on queues, which have also attracted considerable attention to the researchers. For example, Chen and Renshaw (1997) considered queueing models which have allowed the possibility of clearing the entire workload. Chen et al. (2010) combined the bulk-arrival and bulk-service queues with state-dependent control either at idle time or at time with empty waiting line, which thus generalizes the Chen and Renshaw (2004) models to make them more relevant and applicable.

In practical phenomena, it is usual that the server may break down. In these situations, a queueing model with unreliable server is more suitable. Many researchers have contributed on queue with unreliable servers and some noteworthy works are Wang (1995), Wang (1997), and Ke (2005). Maragathasundari et al. (2022) considered and analyzed a single server non-Markovian queue with phase type service and multiple vacation. Deepa and Azhagappan (2022) studied a batch arrival non-Markovian queue with multiple vacation and optional services. Ikhlef et al. (2023) analyzed in a non- Markovian queue with retrials, vacation and server timeout using a Petri net formulation. Laxmi and George (2023) studied a non-Markovian queue with batch service with second optional service. Nagarajan and Kalyanaraman (2023) analyzed in steady state, a non-Markovian queue with batch arrival and batch service with breakdown. Radha et.al. (2023) studied a batch arrival queue with general service time, phase service and multiple vacation. Ayyappan and Nithya (2024) analyzed a single server two types of batch arrivals with retrials, priority services, breakdown, delayed repair, feedback, balking and working vacation.

The following model has been defined and analyzed in this article: A compound Poisson arrival process, fixed batch service queue with single server and with server breakdown, compulsory service vacation and state dependent arrival rates. The single server provides compulsory essential service (ES) and non-compulsory optional service (OS). During busy, the server may breakdown, immediately, it is sent for repair.

The model definition has been introduced in section 2. The steady state analysis has been given in section 3. Some system measures and some particular models are presented in section 4 and 5 respectively. A numerical study is presented in section 6. A conclusion is given in section 7.

2. The Mathematical model

In this section, the model has been defined and relevant notations are introduced. The model definition follows:

- The arrivals are in batches of size X , a random variable X, with probability distribution $\Pr\{\mathrm{X} = \mathrm{j}\} = C_j$, $\mathrm{j} = 1, 2, 3, \dots$ The batch arrivals follows Poisson with state dependent rates.
- There is a single server.
- The server provides service to fixed batch of customers of size K. There are two types of services, called ES and OS. After completing ES, the batch OS with probability q $(0 \leq q \leq 1)$ or leaves the system with may demand probability $(1 - q)$. The service periods S_1(ES) and S_2 (OS) are generally distributed with distribution functions $G_1(x)$ and $G_2(x)$ respectively. The total service time of a batch is $S = (1 - q)S_1 + qS_2$.
- There is a queue with infinite capacity, in which the arriving customers may wait

for their turn.

- After completion of each service, the server takes vacation of compulsory type, the random vacation period V follows general distribution with distribution function is $B(x)$.
- While server is busy, the server may breakdown, the breakdown period follows negative exponential with mean $\frac{1}{\alpha}$. Immediately, the server undergoes repair process, the repair period R, follows a general distribution $H(x)$.
- The arrival rate is

$$\lambda = \begin{cases} \lambda_0 \text{ ,the arrival is during idle period} \\ \lambda_1 \text{ ,the arrival is during essential period} \\ \lambda_2 \text{ ,the arrival is during optional period} \\ \lambda_3 \text{ ,the arrival is during vacation period} \\ \lambda_4 \text{ ,the arrival is during repair period} \end{cases}$$

- The mean batch size, the mean total service period, the mean vacation period and mean repair period are respectively$\mathrm{E(X)}, \mathrm{E(S)}, \mathrm{E(V\,)}$ and $E(R)$.

The hazard function for the distribution function of the essential service time distribution $G_1(x)$ is $\mu_1(x) = \frac{g_1(x)}{1-G_1(x)}$ where $\mu_1(x)\mathrm{dx} = \Pr\{$Essential service will be completed in $(x, x + dx)$ given that the service time exceeds $x\}$

and the hazard function for the distribution function of the optional service time distribution $G_2(x)$ is $\mu_2(x) = \frac{g_2(x)}{1-G_2(x)}$ where $\mu_2(x)\mathrm{dx} = \Pr\{$Optional service will be completed in $(x, x + dx)$ given that the service time exceeds $x\}$.

The hazard function for the distribution function of the vacation time distribution $B(x)$ is $\beta(x) = \frac{b(x)}{1-B(x)}$ where $\beta(x)\mathrm{dx} = \Pr\{$Vacation will be completed in $(x, x + dx)$ given that the vacation time exceeds $x\}$.

The hazard function for the distribution function of the repair time distribution $H(x)$ is $\gamma(x) = \frac{h(x)}{1-H(x)}$ where $\gamma(x)\mathrm{dx} = \Pr\{$Repair will be completed in $(x, x + dx)$ given that the repair time exceeds $x\}$.

At time t, let $K(t)$ be the number of customers in the waiting line and $\eta(t)$ be the supplementary variable at time t. The $\eta(t)$ have the following random identifications.

$$\eta(t) = \begin{cases} \eta_0(t) \text{, the elapsed essential service period} \\ \eta_1(t) \text{ , the elapsed optional service period} \\ \eta_2(t) \text{ , the elapsed vacation period} \\ \eta_3(t) \text{ , the elapsed repair period} \end{cases}$$

The two-dimensional process $\{(K(t), \eta(t)) : t \geq 0\}$ is a Markov process. The following probabilities and probability generating functions are introduced for the analysis:

$$Q_n(t) = \Pr\{K(t) = n, \text{the server is idle}\}, n = 0,1, \dots, K - 1.$$

$$P_{n1}(x;t) = \Pr\{K(t) = n, \eta_0(t) \in (x, x+\Delta t)\}, n = 0,1, \ldots.$$

$$P_{n2}(x;t) = \Pr\{K(t) = n, \eta_1(t) \in (x, x+\Delta t)\}, n = 0,1, \ldots.$$

$$V_n(x;t) = \Pr\{K(t) = n, \eta_2(t) \in (x, x+\Delta t)\}, n = 0,1, \ldots.$$

$$R_n(x;t) = \Pr\{K(t) = n, \eta_3(t) \in (x, x+\Delta t)\}, n = 0,1, \ldots.$$

In steady state,

$$P_{ni}(x) = \lim_{n\to\infty} P_{ni}(x;t)\,; i = 1,2 \;;\; V_n(x) = \lim_{n\to\infty} V_n(x;t),\; R_n(x) = \lim_{n\to\infty} R_n(x;t),$$

$$C(z) = \sum_{j=1}^{\infty} C_j z^j\,, P_i(x,z) = \sum_{n=0}^{\infty} P_{ni}(x) z^n\,; i = 1,2\,, V(x,z) = \sum_{n=0}^{\infty} V_n(x) z^n,$$

$$R(x,z) = \sum_{n=0}^{\infty} R_n(x) z^n\,, Q(z) = \sum_{n=0}^{K-1} Q_n z^n; \text{ where } |z| \le 1$$

3. The Steady state analysis

The system discussed $M^{[X]}/G^K/1$, the following differential-difference equations are obtained using the supplementary variable technique as outlined in Cox (1965).

$$\frac{dP_{01}(x)}{dx} = -(\lambda_1 + \mu_1(x) + \alpha) P_{01}(x) \tag{3.1a}$$

$$\frac{dP_{n1}(x)}{dx} = -(\lambda_1 + \mu_1(x) + \alpha) P_{n1}(x) + \lambda_1 \sum_{j=1}^{n} C_j P_{n-j1}(x)\,, n \ge 1 \tag{3.1b}$$

$$\frac{dP_{02}(x)}{dx} = -(\lambda_2 + \mu_2(x) + \alpha) P_{02}(x) \tag{3.2a}$$

$$\frac{dP_{n2}(x)}{dx} = -(\lambda_2 + \mu_2(x) + \alpha) P_{n2}(x) + \lambda_2 \sum_{j=1}^{n} C_j P_{n-j2}(x)\,, n \ge 1 \tag{3.2b}$$

$$\frac{dV_0(x)}{dx} = -\big(\lambda_3 + \beta(x)\big) V_0(x) \tag{3.3a}$$

$$\frac{dV_n(x)}{dx} = -\big(\lambda_3 + \beta(x)\big) V_n(x) + \lambda_3 \sum_{j=1}^{n} C_j V_{n-j}(x)\,, n \ge 1 \tag{3.3b}$$

$$\frac{dR_0(x)}{dx} = -\big(\lambda_4 + \gamma(x)\big) R_0(x) \tag{3.4a}$$

$$\frac{dR_n(x)}{dx} = -\big(\lambda_4 + \gamma(x)\big) R_n(x) + \lambda_4 \sum_{j=1}^{n} C_j R_{n-j}(x)\,, n \ge 1 \tag{3.4b}$$

$$0 = -\lambda_0 Q_n + \lambda_0\big(1 - \delta_{n,0}\big) \sum_{j=1}^{n} C_j Q_{n-j} + \int_0^{\infty} \gamma(x)\, R_n(x) dx \;+ \int_0^{\infty} \beta(x) V_n(x) dx;\;\; n = 0,1, \ldots, K-1 \tag{3.5}$$

The boundary conditions are,

$$P_{n1}(0) = \int_0^{\infty} \beta(x) V_{n+K}(x) dx + \int_0^{\infty} \gamma(x) R_{n+K}(x) dx + \lambda_0 \sum_{j=0}^{K-1} C_{n+K-j} Q_j\,;\;\; n \ge 0 \tag{3.6a}$$

$$P_{n2}(0) = \int_0^\infty \mu_1(x) P_{n+K1}(x)dx \qquad (3.6b)$$

$$V_n(0) = (1-q)\int_0^\infty \mu_1(x)P_{n+K1}(x)dx + \int_0^\infty \mu_2(x)P_{n+K2}(x)dx; \quad n \geq 0 \qquad (3.7)$$

$$R_n(0) = \alpha\left[(1-q)\int_0^\infty P_{n-K1}(x)dx + \int_0^\infty P_{n+K2}(x)dx\right]; \; n \geq K \qquad (3.8a)$$

$$R_n(0) = 0; n < K \qquad (3.8b)$$

and the normalization condition is

$$\sum_{n=0}^{K-1} Q_n + \int_0^\infty \sum_{n=0}^{\infty} [P_{n1}(x) + P_{n2}(x) + V_n(x) + R_n(x)] \; dx = 1 \qquad (3.9)$$

Theorem 3.1:

Under steady state condition, the model has the following probability generating functions.

$$P_1(z) = \frac{\lambda_0\,[1-C(z)]Q(z)a_2 z^K[1-G_1^*(a_1)]}{J}$$

$$P_2(z) = \frac{\lambda_0\,[1-C(z)]Q(z)a_1 G_1^*(a_1)[1-G_2^*(a_2)]}{J}$$

$$V(z) = \frac{\lambda_0\,[1-C(z)]Q(z)a_1 a_2 G_1^*(a_1)[1-B^*(m_1)]\{z^K(1-q)+G_2^*(a_2)\}}{Jm_1}$$

$$R(z) = \frac{\alpha z^K \lambda_0\,[1-C(z)]Q(z)[1-H^*(m_2)]e_2}{Jm_2}$$

where, $e_2 = \{z^K(1-q)a_2[1-G_1^*(a_1)] + a_1 G_1^*(a_1)[1-G_2^*(a_2)]\}$

$$J = \{z^K(1-q)a_2[1-G_1^*(a_1)] + a_1 G_1^*(a_1)[1-G_2^*(a_2)]\}\alpha z^K$$
$$H^*(m_2) - a_1 a_2[z^{2K} - B^*(m_1)G_1^*(a_1)\{z^K(1-q)+G_2^*(a_2)\}]$$

respectively, the probability generating function of number of customers in queue when the server provides ES, when the server provides OS, when the server provides is on vacation, when the server provides is in breakdown.

Proof:

Multiplication of equations (3.1a) and (3.1b) by appropriate powers of z and adding the resultant equations for $n = 0,1,\ldots\infty$, leads to

$$\frac{\partial}{\partial x}\big(P_1(x,z)\big) + (\lambda_1 - \lambda_1 C(z) + \mu_1(x) + \alpha)\,P_1(x,z) = 0 \qquad (3.10)$$

Multiplication of equations (3.2a) and (3.2b) by appropriate powers of z and adding the resultant equations for $n = 0,1,\ldots\infty$, leads to

$$\frac{\partial}{\partial x}\big(P_2(x,z)\big) + (\lambda_2 - \lambda_2 C(z) + \mu_2(x) + \alpha)\,P_2(x,z) = 0 \qquad (3.11)$$

Multiplication of equations (3.3a) and (3.3b) by appropriate powers of z and adding the resultant equations for $n = 0,1,\ldots\infty$, leads to

$$\frac{\partial}{\partial x}\big(V(x,z)\big) + \big(\lambda_3 - \lambda_3 C(z) + \beta(x)\big)V(x,z) = 0 \tag{3.12}$$

Multiplication of equations (3.4a) and (3.4b) by appropriate powers of z and adding the resultant equations for $n = 0,1,\ldots\infty$, leads to

$$\frac{\partial}{\partial x}\big(R(x,z)\big) + \big(\lambda_4 - \lambda_4 C(z) + \gamma(x)\big)R(x,z) = 0 \tag{3.13}$$

Multiplication of equation (3.6a) by z^{n+K} and summation over $n = 0,1,\ldots\infty$, leads to

$$z^K P_1(0,z) = \int_0^\infty \beta(x) \sum_{n=K}^\infty V_n(x) z^n \, dx + \int_0^\infty \gamma(x) \sum_{n=K}^\infty R_n(x) z^n \, dx + K(z) \tag{3.14}$$

where, $K(z) = \sum_{j=0}^{K-1} \sum_{n=0}^{\infty} z^{n+K} C_{n+K-j} Q_j$

Multiplication of equation (3.5) by z^n and summation over $n = 0,1,\ldots K-1$, leads to

$$0 = -\lambda_0 Q(z) + L(z) + \int_0^\infty \gamma(x) \sum_{n=0}^{K-1} R_n(x) z^n dx + \int_0^\infty \beta(x) \sum_{n=0}^{K-1} V_n(x) z^n dx \tag{3.15}$$

where, $L(z) = \lambda_0\big(1 - \delta_{n,0}\big) \sum_{j=1}^{n} \sum_{n=0}^{K-1} z^n C_j Q_{n-j}$

Multiplication of equation (3.6b) by z^{n+K} and summation over $n = 0,1,\ldots\infty$, leads to

$$P_2(0,z) = \frac{\int_0^\infty P_1(x,z)\mu_1(x)dx}{z^K} \tag{3.16 }$$

Now, addition of equations (3.14) and (3.15), we have

$$z^K P_1(0,z) = \int_0^\infty V(x,z)\beta(x)dx + \int_0^\infty R(x,z)\gamma(x)dx + \lambda_0[C(z)-1]Q(z) \tag{3.17}$$

Multiplication of equation (3.7) by z^n and summation over $n = 0,1,\ldots\infty$, leads to

$$V(0,z) = (1-q)\int_0^\infty P_1(x,z)\mu_1(x)dx + \int_0^\infty P_2(x,z)\mu_2(x)dx \tag{3.18}$$

Multiplication of equations (3.8a) and (3.8b) by appropriate powers of z and adding the resultant equations for $n = 0,1,\ldots\infty$, leads to

$$R(0,z) = \alpha\, z^K[(1-q)P_1(z) + P_2(z)] \tag{3.19}$$

Integrating of equation (3.10) leads to,

$$P_1(x,z) = P_1(0,z)e^{-a_1 x - \int_0^\infty \mu_1(x)dx} \tag{3.20}$$

where, $a_1 = \lambda_1 - \lambda_1 C(z) + \alpha$

Integrating of equation (3.20) leads to,

$$\int_0^\infty P_1(x,z)dx = P_1(z) = \frac{P_1(0,z)\big[1 - G_1^*(a_1)\big]}{a_1} \tag{3.21}$$

Multiplying of equation (3.20) by $\mu_1(x)$ and integration of the equation leads to,

$$\int_0^\infty P_1(x,z)\mu_1(x)dx = P_1(0,z)G_1^*(a_1) \tag{3.22}$$

Integrating of equation (3.11) leads to,

$$P_2(x,z) = P_2(0,z)e^{-a_2 x - \int_0^\infty \mu_2(x)dx} \tag{3.23}$$

where, $a_2 = \lambda_2 - \lambda_2 C(z) + \alpha$

Integrating of equation (3.23) leads to,

$$\int_0^\infty P_2(x,z)dx = P_2(z) = \frac{P_2(0,z)\big[1 - G_2^*(a_2)\big]}{a_2} \tag{3.24}$$

Multiplying of equation (3.23) by $\mu_2(x)$ and integration of the equation leads to,

$$\int_0^{\infty} P_2(x,z)\mu_2(x)dx = P_2(0,z)G_2^*(a_2) \tag{3.25}$$

Substituting the value of equation (3.22), (3.25) in (3.18), we have

$$V(0,z) = (1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2) \tag{3.26}$$

Integrating of equation (3.12) leads to,

$$V(x,z) = V(0,z)e^{-m_1x-\int_0^{\infty}\beta(x)dx} \tag{3.27}$$

where, $m_1 = \lambda_3 - \lambda_3 C(z)$

Substituting the value of equation (3.26) in (3.27), we have

$$V(x,z) = e^{-m_1x-\int_0^{\infty}\beta(x)dx}\{\,(1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2)\,\} \tag{3.28}$$

Integrating of equation (3.28) leads to,

$$\int_0^{\infty} V(x,z)dx = V(z) = \frac{\{\,(1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2)\,\}[1 - B^*(m_1)]}{m_1} \tag{3.29}$$

Multiplying of equation (3.28) by $\beta(x)$ and integration of the equation leads to,

$$\int_0^{\infty} V(x,z)\beta(x)dx = \{\,(1-q)\, P_1(0,z)G_1^*(a_1) + P_2(0,z)G_2^*(a_2)\,\}B^*(m_1) \tag{3.30}$$

Integrating of equation (3.13) leads to,

$$R(x,z) = R(0,z)e^{-m_2x-\int_0^{\infty}\gamma(x)dx} \tag{3.31}$$

where, $m_2 = \lambda_4 - \lambda_4 C(z)$

Substituting the value of equation (3.19), (3.21), (3.24) in (3.31), we have

$$R(x,z) = \frac{\alpha\, z^K e_1\, e^{-m_2x-\int_0^{\infty}\gamma(x)dx}}{a_1a_2} \tag{3.32}$$

where, $e_1 = [(1-q)\, P_1(0,z)[1 - G_1^*(a_1)] + P_2(0,z)[1 - G_2^*(a_2)]]$

Integrating of equation (3.32) leads to,

$$\int_0^{\infty} R(x,z)dx = R(z) = \frac{\alpha\, z^K e_1[1 - H^*(m_2)]}{a_1a_2m_2} \tag{3.33}$$

Multiplying of equation (3.32) by $\gamma(x)$ and integration of the equation leads to,

$$\int_0^{\infty} R(x,z)\gamma(x)dx = \frac{\alpha\, z^K H^*(m_2)e_1}{a_1a_2} \tag{3.34}$$

Substituting the value of equation (3.22) in (3.16), we have

$$P_2(0,z) = \frac{P_1(0,z)G_1^*(a_1)}{z^K} \tag{3.35}$$

Substituting the value of equation (3.22), (3.25), (3.30), (3.34), (3.35) in (3.17), we have

$$P_1(0,z) = \frac{\lambda_0\,[C(z)-1]Q(z)a_1a_2z^K}{O} \tag{3.36}$$

where,

$$O = a_1a_2[z^{2K} - B^*(m_1)G_1^*(a_1)\{z^K(1-q) + G_2^*(a_2)\} - \{z^K(1-q)a_2[1 - G_1^*(a_1)] + a_1G_1^*(a_1)[1 - G_2^*(a_2)]\}\alpha z^K H^*(m_2)$$

Substituting the value of equation (3.36) in (3.35), we have

$$P_2(0,z) = \frac{\lambda_0\,[C(z)-1]Q(z)a_1a_2G_1^*(a_1)}{O} \tag{3.37}$$

Substituting the value of equation (3.36) in (3.21), we have

$$P_1(z) = \frac{\lambda_0\,[1-C(z)]Q(z)a_2z^K[1-G_1^*(a_1)]}{J} \tag{3.38}$$

Substituting the value of equation (3.37) in (3.24), we have

$$P_2(z) = \frac{\lambda_0\,[1-C(z)]Q(z)a_1G_1^*(a_1)[1-G_2^*(a_2)]}{J} \tag{3.39}$$

Substituting the value of equation (3.36) and (3.37) in (3.29), we have

$$V(z) = \frac{\lambda_0\,[1-C(z)]Q(z)a_1a_2G_1^*(a_1)[1-B^*(m_1)]\{z^K(1-q)+G_2^*(a_2)\}}{Jm_1} \tag{3.40}$$

Substituting the value of equation (3.36) and (3.37) in (3.33), we have

$$R(z) = \frac{\alpha z^K\lambda_0\,[1-C(z)]Q(z)[1-H^*(m_2)]e_2}{Jm_2} \tag{3.41}$$

Theorem 3.2:

Under the steady state condition, the probability generating function for number of customers in the queue is $S(z)$, where $S(z) = Q(z) + N(z)$.

Proof:

Let $S(z) = Q(z) + P_1(z) + P_2(z) + V(z) + R(z)$ (3.42)

$$N(z) = P_1(z) + P_2(z) + V(z) + R(z)$$

Now, Substituting the value of equation (3.38), (3.39), (3.40), (3.41) in (3.42), we have

$$S(z) = \frac{Q(z)X}{Jm_1m_2} \tag{3.43}$$

where, $X = m\{a_2z^K[1-G_1^*(a_1)]m_1m_2 + m_1m_2a_1G_1^*(a_1)[1-G_2^*(a_2)] + m_2a_1a_2G_1^*(a_1)\,[1-B^*(m_1)]\{z^K(1-q)+G_2^*(a_2)\} + \alpha z^K[1-H^*(m_2)]e_2m_1\} + Jm_1m_2;\, m = \lambda_0 - \lambda_0C(z)$

$S(z) = \frac{A}{z-z_0}$ by substituting $z = 1$, we get

$$A = (1-z_0)S(1)$$

$$S(1) = \frac{Q(f_1+f_2)}{f_2} \tag{3.44}$$

where, $f_1 = \lambda_0E(X)\{\,[1-G_1^*(\alpha)] + G_1^*(\alpha)[1-G_2^*(\alpha)] + \alpha E(R)\{[1-G_1^*(\alpha)](1-q) + G_1^*(\alpha)[1-G_2^*(\alpha)] + \alpha E(V)G_1^*(\alpha)[(1-q)+G_2^*(\alpha)]\}\}$

$f_2 = \alpha G_1^*(\alpha)[(1-q)+G_2^*(\alpha)][K - E(X)E(V)\lambda_3] - \lambda_1E(X)[1-G_1^*(\alpha)](1-q) - \lambda_1E(X)[1-G_2^*(\alpha)G_1^*(\alpha)] - \alpha\lambda_4E(X)E(R)\{[1-G_1^*(\alpha)](1-q) + G_1^*(\alpha)[1-G_2^*(\alpha)]\}$

By applying Rouche's theorem,

Substituting the value of $S(1)$ in the above equation, we have

$$A = \frac{(1-z_0)Q(f_1+f_2)}{f_2} \tag{3.45}$$

Substituting the value of (3.45) in the above equation, we have

$$S(z) = \frac{(z_0-1)Q(f_1+f_2)}{z_0f_2}\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n \tag{3.46}$$

Which is the probability generating function of number of customers in the queue.

4. System measures

In this section, to shows the performance of the model, the following system measures are obtained:

1. The idle probability is $Q = \sum_{n=0}^{K-1} Q_n$ which leads to, $Q = \frac{(f_2 - f_1)}{f_2}$ (4.1)

   This is obtained by using $Q + N(1) = 1 \Rightarrow Q = 1 - N(1)$

2. The average number of customers in the queue when the server provides ES

$$N_1 = P_1'(1) = \frac{f_3}{4f_2} \tag{4.2}$$

where, $f_3 = Q\lambda_0 E(X)\Big[\lambda_0 E(X)\,[1 - G_1^*(\alpha)] + 4\lambda_2 E(X)\,[1 - G_1^*(\alpha)] - [1 - G_1^*(\alpha)] - 4K[1 - G_1^*(\alpha)] - 4\left[\lambda_1 E(X) G_1^{*'}(\alpha)\right]\Big]$

3. The average number of customers in the queue when the server provides OS

$$N_2 = P_2'(1) = \frac{f_4}{f_2} \tag{4.3}$$

where, $f_4 = Q\lambda_0 E(X)^2 \alpha G_1^*(\alpha)[1 - G_2^*(\alpha)] - \lambda_0 E(X) Q\,[1 - G_1^*(\alpha)] G_2^*(\alpha) + E(X)\,\lambda_1 E(X) Q [1 - G_2^*(\alpha)] G_1^*(\alpha) - 4\lambda_0 E(X) Q \lambda_1 E(X)\, G_1^{*'}(\alpha)[1 - G_2^*(\alpha)] + 4\lambda_0 E(X) Q\,\left[\lambda_2 E(X) G_2^{*'}(\alpha)\right] G_1^*(\alpha)$

4. The average number of customers in the queue when the server provides compulsory vacation

$$N_3 = V'(1) = \frac{\lambda_0 E(X) Q f_5}{24\lambda_3 E(X)} \tag{4.4}$$

where, $f_5 = \alpha^2(\lambda_3 E(X)^2 - \lambda_3 E(X)) E(V) G_1^*(\alpha)[(1-q) + G_2^*(\alpha)] + 4\lambda_1 (E(X))^2\, \lambda_3 \alpha E(V) G_1^*(\alpha)[(1-q) + G_2^*(\alpha)] + 4\lambda_2 E(X)\lambda_3 E(X)\alpha E(V) G_1^*(\alpha)\, [(1-q) + G_2^*(\alpha)] + 4\lambda_1 E(X) G_1^{*'}(\alpha)(\lambda_3 E(X)\alpha E(V) + (\lambda_3 E(X)^2\, B^{*'}(m_1))[(1-q) + G_2^*(\alpha)]\alpha^2 - 4\lambda_1 E(X) B^{*'}(m_1)(\lambda_3 E(X))\alpha^2\, G_1^*(\alpha) - \alpha^2 G_1^*(\alpha)(\lambda_3 E(X)^2 E(V)[(1-q)K + G_2^*(\alpha)] + [(1-q)K + \lambda_2 E(X) G_2^{*'}(\alpha)]$

5. The average number of customers in the queue when the server provides repair

$$N_4 = R'(1) = \frac{f_6}{12\lambda_4 E(X)} \tag{4.5}$$

where, $f_6 = Q\lambda_0 E(X)\{2(\lambda_4 E(X)\alpha E(R)((1-q)[1 - G_1^*(\alpha)]\lambda_2 E(X) + 2\alpha G_1^*(\alpha)\, [1 - G_2^*(\alpha)]\} + 2(\lambda_4 E(X)\alpha E(R)\{(1-q)[1 - G_1^*(\alpha)] + \lambda_1 E(X) G_1^*(\alpha)[1 - G_2^*(\alpha)]\} - 2\alpha^2\{(1-q)[1 - G_1^*(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)]\})\{(\lambda_4 E(X) H^{*'}(m_2) + (\lambda_4 E(X)^2\, H^{*''}(m_2)\}) - 2\alpha^2\{(1-q)K[1 - G_1^*(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)]\} - 2\alpha^2\lambda_4 E(X) E(R)\, \{(1-q)[\lambda_1 E(X) G_1^{*'}(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)]\} - 2\alpha^2\lambda_4 E(X) E(R)\{(1-q)\, [1 - G_1^*(\alpha)] + G_1^*(\alpha)[\lambda_2 E(X) G_2^{*'}(\alpha)]\} - \alpha^2(\lambda_0 E(X) - \lambda_0 E(X)^2)\lambda_4 E(X) E(R)\, Q\{(1-q)[1 - G_1^*(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)]\}$

6. The average number of customers in the queue

$$S = S'(1) = \frac{Q(f_1 + f_2)}{(1 - z_0) f_2} \tag{4.6}$$

7. The server's utilization factor is
$$\rho = 1 - Q \quad (4.7)$$
8. Mean waiting time of a customer
$$W = \frac{S}{\lambda^*} \quad (4.8)$$
Where, λ^* is the effective arrival rate and
$$\lambda^* = Q\lambda_0 + P_1(1)\lambda_1 + P_2(1)\lambda_2 + V(1)\lambda_3 + R(1)\lambda_4$$

5. Particular Models

In this section, some particular models are obtained by taking particular form to the distribution functions and particular values to the parameters.

Particular model 1:

The model in this paper is particularized by removing a OS ($q = 0$) and taking state independent parameters (($\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$) the probability generating function of numbers of customers in the queue is $S(z) = \frac{D_1}{D_2}$.

where, $D_1 = Q(z)[\{m + H^*(m)\alpha z^K\}\{[1 - G^*(a)] + aG^*(a)[1 - B^*(m)] + D_2\}$

$$D_2 = H^*(m)\alpha z^K[1 - G^*(a)] - a[z^K - B^*(m)G^*(a)]$$

Particular model 2:

The model in this paper is particularized by taking state independent parameters (($\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$) the probability generating function of numbers of customers in the queue is $S(z) = \frac{D_3}{D_4}$.

where, $D_3 = Q(z)[m\{z^K[1 - G_1^*(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)] + \alpha z^K[1 - H^*(m)]\{(1 - q)z^K[1 - G_1^*(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)]\} + \alpha G_1^*(\alpha)[1 - B^*(m)]\{(1 - q)z^K + G_2^*(\alpha)\}\} + D_4$

$$D_4 = H^*(m)\alpha z^K\{(1 - q)z^K[1 - G_1^*(\alpha)] + G_1^*(\alpha)[1 - G_2^*(\alpha)]\} - a[z^{2K} - B^*(m)G^*(a)\{(1 - q)z^K + G_2^*(\alpha)\}$$

6. Numerical Study

In this section, numerical study for the model discussed in this paper is carried out by assuming service times, vacation times and repair time as negative exponential distribution and batch size as geometric distribution, The parameter values are $C_j = \delta(1 - \delta)^{j-1}$, $j = 1,2, \ldots; 0 < \delta < 1$.
$$E(X) = \frac{(1 - \delta)}{\delta}, E(V) = \frac{1}{\beta}, E(R) = \frac{1}{\gamma}, G_1^*(\alpha) = \frac{\mu_1}{\alpha + \mu_1}, G_2^*(\alpha) = \frac{\mu_2}{\alpha + \mu_2},$$
$$B^*(m_1) = \frac{\beta}{\beta + m_1}, H^*(m_2) = \frac{\gamma}{\gamma + m_2}, E(S_1) = \frac{1}{\mu_1}, E(S_2) = \frac{1}{\mu_2}$$
The performance measures are calculated using the formulas in section 4.

1. The idle probability is $Q = \frac{(f_8 - f_7)}{f_8}$
   where, $f_7 = \lambda_0(1-\delta)\alpha(\alpha+\mu_2)\gamma\beta + \lambda_0(1-\delta)\alpha\mu_1\gamma\beta + \alpha\beta\lambda_0(1-\delta)$
   $[\alpha(\alpha+\mu_2)(1-q)+\mu_1\alpha] + \lambda_0\gamma\alpha(1-\delta)[\mu_1(\alpha+\mu_2)(1-q)+\mu_1\mu_2]$
   $f_8 = \gamma\alpha[\mu_1(\alpha+\mu_2)(1-q)+\mu_1\mu_2][\delta\beta K - \lambda_3(1-\delta)] - \lambda_1\alpha\gamma\beta(1-\delta)(1-q)$
   $-\lambda_2\gamma\beta(1-\delta)(\alpha^2+\alpha\mu_1+\alpha\mu_2) - \alpha\beta\lambda_4(1-\delta)[\alpha(\alpha+\mu_2)(1-q)+\mu_1\alpha]$
2. The average number of customers in the queue when the server provides ES
   $N_1 = P_1'(1) = \frac{Q\lambda_0 f_9}{4\alpha\delta\mu_1 f_8}$
   where, $f_9 = \left(2-3\delta+\delta^2\right) - \delta(1-\delta)\mu_1\alpha + 4\lambda_2\alpha(1-\delta)^2 + 4K\alpha\mu_1\delta(1-\delta)$
   $+4\lambda_1(1-\delta)(\alpha+\mu_2)(\alpha+\mu_1)$
3. The average number of customers in the queue when the server provides OS
   $N_2 = P_2'(1) = \frac{Q\beta\gamma\lambda_0 f_{10}}{4\alpha\mu_2\mu_1 f_8}$
   where, $f_{10} = \left(2-3\delta+\delta^2\right) - \delta(1-\delta)\,\alpha^3(\mu_1)^2\mu_2 - (\alpha\mu_1)^2\mu_2\delta(1-\delta) +$
   $4\lambda_1(1-\delta)\alpha\mu_2(\mu_1)^2 - 4[\lambda_1\mu_2 - \mu_1\lambda_2](1-\delta)(\alpha+\mu_2)(\alpha+\mu_1)$
4. The average number of customers in the queue when the server is on compulsory vacation

   $N_3 = V'(1) = \frac{f_{11}}{24(\alpha+\mu_2)(\alpha+\mu_1)\mu_2\delta^2\beta^2\mu_1}$

   where, $f_{11} = \lambda_0(1-\delta)Q\{[\lambda_3(1-\delta)\mu_1\mu_2\beta\delta((1-q)(\alpha+\mu_2)+\mu_2)][4\mu_2(\mu_1)^2$
   $\alpha^2\delta(\alpha+\mu_2)(\alpha+\mu_1) - 4\mu_1\mu_2\beta(\alpha\delta)^2(\alpha+\mu_2) + 16(1-\delta)(\mu_1)^2\delta\mu_2[\lambda_1(\alpha+\mu_2)\delta$
   $+\lambda_2(\alpha+\mu_1)] - 16\alpha^2\mu_2[(1-q)(\alpha+\mu_2)+\mu_2]\lambda_1(\alpha+\mu_2)\delta(1-\delta)]\} - 4\mu_1\mu_2\beta\alpha^2$
   $[-\lambda_3(\delta(1-\delta)+(2-3\delta+\delta^2)) + 2\lambda_3(2-3\delta+\delta^2)][\mu_2(1-q)+\mu_2+(1-\delta)]$
   $+16\alpha^2\lambda_3\mu_1\delta(1-\delta)[K(1-q)(\alpha+\mu_2)+\mu_2] - 16\alpha^2\lambda_3\mu_1\delta(1-\delta)[(1-q)(\alpha+\mu_2)$
   $+\mu_2]\}+4Q\lambda_0(\alpha\mu_1)^2\mu_2\beta\lambda_3(1-\delta)[6\lambda_3(2-3\delta+\delta^2)+24\delta^2]$
5. The average number of customers in the queue when the server is on repair
   $N_4 = R'(1) = \frac{f_{12}}{12\gamma^2\delta^2\mu_1\mu_2(\alpha+\mu_2)(\alpha+\mu_1)}$
   where, $f_{12} = 4Q\lambda_0\{\gamma\lambda_4\left(\left(2-3\delta+\delta^2\right)-\delta(1-\delta)\right) + 2\lambda_4\left(2-3\delta+\delta^2\right)$
   $[\alpha(\alpha+\mu_2)(1-q)+\mu_1\alpha]\mu_2\mu_1\} + 8\alpha\delta Q\,\lambda_0(1-\delta)\gamma\lambda_4[\alpha K(\alpha+\mu_2)(1-q)$
   $+\mu_1\alpha]\mu_1 + \alpha\gamma^2\delta\mu_1[(1-q)\lambda_1(1-\delta)(\alpha+\mu_2)(\alpha+\mu_1)+(\mu_1)^2\alpha\delta] - \alpha\gamma^2$
   $\delta\mu_1[(1-q)\lambda_2(1-\delta)(\alpha+\mu_2)(\alpha+\mu_1)+(\mu_1)^2\alpha\delta] - [\delta\alpha(1-q)(\alpha+\mu_2)$
   $+\lambda_1(1-\delta)\mu_1\alpha]\,\alpha\gamma^2\delta\mu_1 + \alpha\gamma^2\delta(\mu_1)^2[(1-q)\alpha^2(\alpha+\mu_2)+\alpha^3\lambda_1(1-\delta)$
   $(\alpha+\mu_1)]\mu_1 + \gamma^2\delta(\mu_1)^2[(1-q)\alpha^2(\alpha+\mu_2)+\alpha^3\lambda_2(1-\delta)(\alpha+\mu_1)]\mu_1$
   $\alpha^2\gamma\lambda_0\delta[(1-\delta)\delta(2-3\delta+\delta^2)]\lambda_4(1-\delta)[\alpha(\alpha+\mu_2)-(1-q)+\mu_1\alpha]$
   $\mu_1\mu_2 + 4\lambda_0\lambda_4(1-\delta)[(1-\delta)\delta\gamma\mu_1\mu_2[\alpha(\alpha+\mu_2)(1-q)+\alpha\mu_1]$
   $[3\lambda_4(2-3\delta+\delta^2)+9]$
6. The average number of the customers in the queue is
   $S = S'(1) = \frac{Q(f_7+f_8)}{(1-z_0)f_8}$

7. The server's utilization factor is

$$\rho = \frac{f_{13}}{f_{14}}$$

where, $f_{13} = \alpha\beta\gamma(1-\delta)[(1-q)\lambda_1 + \lambda_0(\alpha+\mu_2)] + \beta\gamma(\alpha+\mu_2)$
$[\lambda_2(\alpha^2 + \alpha\mu_1 + \alpha\mu_2) + \lambda_0\mu_1\alpha] + \alpha\gamma\lambda_0(1-\delta)[\mu_1(\alpha+\mu_2)(1-q)$
$+\mu_1\mu_2] + \alpha\beta(1-\delta)[\alpha(\alpha+\mu_2)(1-q) + \mu_1\alpha][\gamma\lambda_0 + \lambda_4]$

$f_{14} = \gamma\alpha[\mu_1(\alpha+\mu_2)(1-q) + \mu_1\mu_2][\delta\beta K - \lambda_3(1-\delta)]$

8. Mean waiting time of a customer

$$W = \frac{S}{\lambda^*}$$

Where, λ^* is the effective arrival rate and

$$\lambda^* = Q\lambda_0 + P_1(1)\lambda_1 + P_2(1)\lambda_2 + V(1)\lambda_3 + R(1)\lambda_4$$

$$P_1(1) = \frac{\lambda_0 Q(1-\delta)\alpha(\alpha+\mu_2)\beta\gamma}{f_8}$$

$$P_2(1) = \frac{\lambda_0 Q(1-\delta)\alpha\mu_1\beta\gamma}{f_8}$$

$$V(1) = \frac{\lambda_0 Q(1-\delta)\alpha\gamma[\mu_1(\alpha+\mu_2)(1-q)+\mu_1\mu_2]}{f_8}$$

$$R(1) = \frac{\lambda_0 Q(1-\delta)\alpha\beta[\alpha(\alpha+\mu_2)(1-q)+\mu_1\alpha]}{f_8}$$

TABLE 1: Performance Measures $\alpha = 0.01, \beta = 1.0, \gamma = 3.9, \delta = 0.4, q = 0.9,$ $z = 0.89, K = 18, \lambda_1 = 2.9, \lambda_2 = 2.5, \lambda_3 = 1.9, \lambda_4 = 1.8, \mu_1 = 1.5, \mu_2 = 1.9$

λ_0	ρ	Q	N_1	N_2	N_3	N_4	S
1.1	0.6918	0.5787	5920.6855	2563.7886	0.0436	0.1034	8.3204
1.2	0.7032	0.4664	4771.8677	2254.1729	0.0936	0.0936	9.4632
1.3	0.7147	0.3854	3943.6907	2017.8533	0.0342	0.0864	10.5715
1.4	0.7262	0.3247	3321.6907	1830.6477	0.0310	0.0809	11.6526
1.5	0.7377	0.2778	2841.8840	1678.0897	0.0284	0.0767	12.7120
1.6	0.7491	0.2407	2462.4749	1550.9913	0.0262	0.0733	13.7537
1.7	0.7606	0.2108	2156.5654	1443.2087	0.0244	0.0706	14.7808
1.8	0.7721	0.1863	1905.8743	1350.4681	0.0288	0.0685	15.7959
1.9	0.7835	0.1659	1697.5756	1269.6975	0.0214	0.0668	16.8007
2.0	0.7950	0.1488	1522.4274	1198.6274	0.02020	0.0654	17.7969

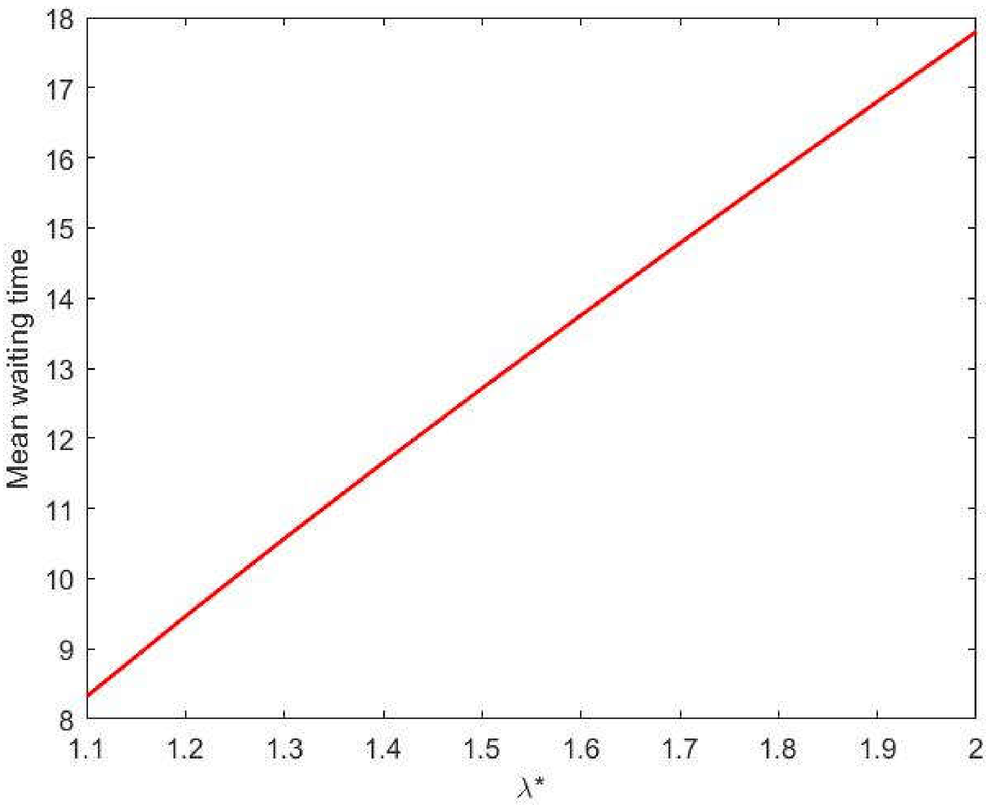

Figure 1 :Mean Waiting time *against* λ^*

The performance measures $\rho, Q, N_1, N_2, N_3, N_4, S$ and W are numerically calculated by changing the parameters values. The results are presented in Table 1 and Figure 1. From the Table 1, shows various mean values corresponding to number of customers in the model by fixing some parameters and varying the arrival rate λ_0. It shows the stability of the formulas derived in the article. Figure 1 shows the mean waiting time against the effective arrival rate λ^*. From the figure, it is clear that as λ^* increases, the mean waiting time also increases.

7. Conclusion

This article deals with $M^{[X]}/G^K/1,$ queue with server breakdown, compulsory server vacation, state dependent arrival rates and two types of services. This model is completely analyzed by defining suitable probability structure and probability generating functions. Numerical result is also presented to show the practical applicability of the model.

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