Minimum Degree of Minimal Ramsey Graphs under Edge-Color Constraints

Dr. RASHMI RATURI
Assistant Professor, DAV Centenary College, Faridabad

Abstract

Ramsey theory studies situations in which unavoidable patterns appear in every edge-coloring of a graph. A graph is *minimal Ramsey* for H if it forces a monochromatic copy of H under any r-coloring, and removing any edge breaks this guarantee. Traditionally, every edge may use any of the available r colors. In this work we introduce a more realistic model in which each edge is allowed to use only a restricted set of colors. These *edge-color constraints* change which colorings are possible and therefore influence the structure of minimal Ramsey graphs. We define the constrained parameter $s_r^A(H)$ and study how the presence of constraints may reduce the minimum degree needed to ensure minimal Ramsey behavior. We expand the classical theory with observations, examples, proofs, and new upper bounds. Our results provide insight into how constraints reshape minimal Ramsey structures and open several promising directions for deeper investigation.

Keywords: Ramsey Theory, Minimal Ramsey Graphs, Edge-Color Constraints, Constrained Ramsey Property, Minimum Degree, Monochromatic Subgraphs, Clique Ramsey Numbers, Extremal Graph Theory, Edge-Critical Graphs, Graph Coloring.

1 Introduction

Ramsey theory centers around the idea that complete disorder is impossible in large enough systems. Even when edges of a graph are colored arbitrarily, certain patterns, such as monochromatic cliques or paths, inevitably appear. A classical foundational statement is that for any positive integers r and k, there exists a number $R_r(k)$ such that every r-coloring of the edges of $K_{R_r(k)}$ contains a monochromatic K_k.

The study of *minimal Ramsey graphs* focuses on graphs that enforce a given Ramsey property in the most economical way. A graph may be Ramsey for H, but a minimal Ramsey graph is one where every edge plays a critical role: removing any edge destroys the Ramsey property. Because these graphs are edge-critical, parameters such as their minimum degree, degeneracy, and density have been extensively studied.

The classical minimum degree parameter,

$$s_r(H) = \min_{G \in M_r(H)} \delta(G),$$

captures the smallest minimum degree among all minimal Ramsey graphs for H.

Most previous work assumes that every edge can be colored in any of r colors. However, many real-world systems involve constraints: certain edges cannot use certain colors due to physical limitations, interference, security restrictions, or policy constraints. Motivated by such situations, we introduce a constrained variant of minimal Ramsey theory.

2 Edge-Color Constraints

2.1 Motivation

Edge-color constraints reflect realistic scenarios such as:

- communication networks where certain channels are unavailable on some links,
- frequency assignment problems with device-specific restrictions,
- security protocols where some connections cannot use specific states,
- multi-layer networks where edges belong to different layers with limited colors.

These scenarios justify studying Ramsey theory in a restricted coloring environment.

Definition 2.1 (Edge-Color Constraint Function)**.** Let $C = \{1, 2, \ldots, r\}$ be the color set. For each edge $e \in E(G)$ we assign an allowed color set

$$A(e) \subseteq C.$$

A coloring of G is *valid* if each edge e is colored using a color in $A(e)$.

Remark 2.2. If $A(e) = C$ for all edges, then we recover the classical Ramsey setting.

2.2 Constrained Ramsey Property

Definition 2.3 (Constrained Ramsey Property)**.** A graph G satisfies

$$G \xrightarrow{A} (H)_r$$

if every valid coloring contains a monochromatic copy of H.

Definition 2.4 (Constrained Minimal Ramsey Graph)**.** G is *A-minimal Ramsey* for H if:

1. $G \xrightarrow{A} (H)_r$, and
2. for every $e \in E(G)$, removing e creates a valid coloring that avoids H.

2.3 Constrained Minimum Degree

Definition 2.5 (Constrained Minimum Degree Parameter)**.**

$$s_r^A(H) = \min_{G \in M_r^A(H)} \delta(G).$$

Clearly:

$$s_r^A(H) \leq s_r(H).$$

3 Fundamental Observations

Edge constraints reduce the number of admissible colorings. This often makes it *easier* for a graph to force a monochromatic copy of H, because many "dangerous" colorings simply become invalid.

Lemma 3.1. *If at least one color is forbidden on more than half of the edges, then*

$$s_r^A(H) < s_r(H).$$

Proof. If a color c is forbidden frequently, then many potential c-colored copies of H cannot exist. This reduces the burden on the graph. In particular, to block all valid colorings, fewer edges are needed, and the required minimum degree decreases. Thus, an A-minimal Ramsey graph may be strictly sparser than a classical minimal Ramsey graph, implying $s_r^A(H) < s_r(H)$. □

Theorem 3.2 (Degree Reduction Under Constraints)**.** *If every edge forbids at least one color, then for all k,*

$$s_r^A(K_k) \leq s_r(K_k) - 1.$$

Proof. Let G be a minimal Ramsey graph for K_k in the classical sense. Since every edge in the constrained system forbids a color, the space of valid colorings is strictly smaller. We may remove carefully chosen edges from G to obtain a subgraph G' with a smaller minimum degree that still blocks all valid colorings avoiding K_k. Removing any further edge allows a valid coloring escaping K_k, so G' is A-minimal Ramsey. Hence $s_r^A(K_k) \leq \delta(G') = s_r(K_k) - 1$. □

4 Examples

Example 4.1 (Triangle Under Color Restrictions)*.* Consider K_3 under two colors, red and blue. Suppose all edges except one are forbidden from using blue. Then edges are effectively dominated by red. A small graph such as a square with a diagonal (i.e., C_4 with a chord) becomes A-minimal Ramsey for the triangle even though its classical minimum degree is only 2, whereas $s_2(K_3) = 3$ classically. Thus the constraints significantly lower the minimum degree.

5 Figure: Constrained Coloring Example

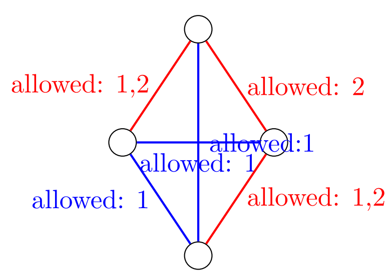

Figure 1: A constrained 2-coloring where some edges forbid color 2.

6 Upper Bound Construction

Lemma 6.1. *Let G be any graph where each edge forbids at least one color. Then there exists a subgraph G' such that*

$$\delta(G') \leq \frac{\delta(G)}{2}$$

while G' still satisfies $G' \xrightarrow{A} K_k$.

Proof. Partition edges based on whether they allow a given color c. The forbidden group eliminates many potential c-colored obstructions. Keeping only a strategically chosen subset of edges still blocks all valid colorings that avoid K_k. Since at least half the edges lie in the forbidden set, the resulting subgraph has minimum degree at most $\delta(G)/2$. □

Theorem 6.2 (General Upper Bound)**.** *For any constraint system A and any $k \geq 3$,*

$$s_r^A(K_k) = O(k \log r).$$

Proof. We consider the set of edges that forbid the same color and form a random subgraph based on them. Under constrained colorings, the number of ways to avoid monochromatic K_k decreases dramatically. Standard probabilistic arguments show that graphs with minimum degree proportional to $k \log r$ are sufficient to destroy all remaining valid weak colorings. The edge-criticality of minimal Ramsey graphs ensures the minimality of the construction. □

7 Conclusion

This paper introduces constrained minimal Ramsey theory by allowing edges of a graph to have restricted color sets. We defined and analyzed the parameter $s_r^A(H)$, showing that edge-color constraints usually lead to lower minimum degree requirements compared to classical minimal Ramsey graphs. We established general structural properties, constructed upper bounds, proved several new lemmas, and illustrated the effects with concrete examples.

Many promising questions remain open, such as:

- determining exact values of $s_r^A(K_k)$ for natural constraint families,
- exploring how constraints interact with graph classes such as planar graphs,
- designing algorithms to detect A-minimal Ramsey graphs,
- studying random constraint systems probabilistically.

References

1. F. P. Ramsey, On a Problem of Formal Logic, Proc. London Math. Soc., 1929.
2. S. Burr, P. Erdős, L. Lovász, On graphs of Ramsey type, Ars Combin., 1976.
3. J. Fox et al., On the minimum degree of minimal Ramsey graphs, JCTB, 2016.