Fuzzy Doubt p-Ideal of Z-Algebra

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ABSTRACT

The Concept of this study is to present the definition of Fuzzy Doubt p-ideals in Z-algebras and investigate some of their properties. Also we found that an homomorphic and Cartesian product of Fuzzy Doubt p-Ideal is also a Fuzzy Doubt p-Ideal. Moreover some characteristics theorem like union, intersection are also explored.

KEYWORDS

Z-algebra, Fuzzy set, Fuzzy Doubt p-ideal, Intersection and Union,.

1. INTRODUCTION

Fuzzy mathematics is the branch of mathematics introduced by Lofti Asker Zadeh [5] including fuzzy set theory and fuzzy logic that deals with partial inclusion of elements in a set to simple binary "yes' or 'no" (0 or 1) inclusion. Fuzzy mathematics has its orgin on fuzzy set introduced by Lofti Asker Zadeh [5]. Fuzzy set theory has been developed in many directions by many scholars and has evolved a great interest among mathematicians working in different fields of mathematics. Y. Imai and K. Iseki [2] proposed BCK- Algebras and BCI- Algebras. Additionally, the BCKalgebra class is a proper subclass of the BCI-algebra class.

In 2017, Chandramouleeswaran [7], defined the concept of Z-algebras. Then in 2020, S. Sowmiya [8] [9] [10] [11] [12] gave the concept of Fuzzy Ideals of Z-algebras. Following the same route, S. Sowmiya established the definition of the Fuzzy Ideals of h-algebras and Fuzzy p-Ideals of Z-algebras [12].

In the last two decades interest of many mathematicians has shifted to the development of Fuzzy algebra in view of generalisation of the well-known rules of algebraic structures. Many mathematicians have been involved in extending the concepts and results of abstract algebra. In this ambitious work, we define a new concept of Fuzzy Doubt p-Ideal [FDpI] in Z-algebras A_Z and to study some of their properties. Also we recall some basic concepts for the sake of completeness

2. PRELIMINARIES

We first list some basic concepts which are needed for our work.

Definition 1. [5] Consider I_ζ be a subset of A_Z, then it is defined as an ideal of A_Z if it satisfies

- $0 \in I_\zeta$,
- $\omega_\zeta * \kappa_\zeta \in I_\zeta$ and $\kappa_\zeta \in I_\zeta$ imply $\omega_\zeta \in I_\zeta$, for all $\omega_\zeta, \kappa_\zeta \in A_Z$".

Definition 2. [5] Consider $(A_\zeta, *, 0)$ and $(A_\zeta', *, 0)$ be a A_Z, then $h_\zeta : (A_\zeta, *, 0) \rightarrow (A_\zeta', *, 0)$ is stated as Z-homomorphism of A_Z if $h_\zeta(\omega_\zeta * \kappa_\zeta) = h_\zeta(\omega_\zeta) * h_\zeta(\kappa_\zeta)$ for all $\omega_\zeta, \kappa_\zeta \in A_Z$.

Definition 3. [9] Let homomorphism of A_Z, $h_\zeta : (A_\zeta, *, 0) \rightarrow (A_\zeta', *, 0)$, then h_ζ is defined as

- If h_ζ is one-one then A_ζ is monomorphism.
- If h_ζ is onto then A_ζ is epimorphism of A_Z.
- If h_ζ is a mapping $(A_\zeta, *, 0)$ into itself, then $A_\zeta$is endomorphism.

Definition 4. [5] Let A_Z is a non-empty set consist of a binary operation $*$ and a constant 0 satisfying the following conditions:

- $\omega_\zeta * 0 = 0$
- $0 * \omega_\zeta = \omega_\zeta$
- $\omega_\zeta * \omega_\zeta = \omega_\zeta$
- $\omega_\zeta * \kappa_\zeta = \kappa_\zeta * \omega_\zeta$ when $\omega_\zeta \neq 0$ and $\kappa_\zeta \neq 0$ for every $\omega_\zeta, \kappa_\zeta \in A_Z$.

Definition 5. [6] A fuzzy set δ of A_Z is called a fuzzy z- ideal of A_Z if it satisfies

- $\delta_\zeta(0) \geq \delta_\zeta(\omega_\zeta)$
- $\delta_\zeta(\omega_\zeta) \geq \{\delta_\zeta(\omega_\zeta * \kappa_\zeta) \wedge \delta_\zeta(\kappa_\zeta)\}$, for all $\omega_\zeta, \kappa_\zeta \in A_Z$.

Definition 6. [8] A Fuzzy Set δ of A_Z is called a Fuzzy h-Ideal of A_ζ if it satisfies

- $\delta_\zeta(0) \geq \delta_\zeta(\omega_\zeta)$
- $\delta_\zeta(\omega_\zeta * \eta_\zeta) \geq \{\delta_\zeta(\omega * (\kappa_\zeta * \eta_\zeta)) \wedge \delta_\zeta(\kappa_\zeta)\}$, for all $\omega_\zeta, \kappa_\zeta \in A_Z$.

Definition 7. [7] A Fuzzy Set δ of A_Z is called a Fuzzy p-Ideal of A_ζ if it satisfies

- $\delta_\zeta(0) \geq \delta_\zeta(\omega_\zeta)$
- $\delta_\zeta(\omega_\zeta * \eta_\zeta) \geq \{\delta_\zeta((\omega_\zeta * \kappa_\zeta) * (\eta_\zeta * \kappa_\zeta)) \wedge \delta_\zeta(\kappa_\zeta)\}$, for all $\omega_\zeta, \kappa_\zeta \in A_Z$.

Definition 8. [8] Let A_Z be a medial if it satisfies $(\omega_\zeta * (\omega_\zeta * \kappa_\zeta)) = \kappa_\zeta$, for all $\omega_\zeta, \kappa_\zeta \in A_Z$.

Throughout this work

- A_Z means Z-algebra,
- FS means Fuzzy set,
- FDzI means Fuzzy Doubt z-Ideal,
- FDpI means Fuzzy Doubt p-Ideal,
- FDhI means Fuzzy Doubt h-Ideal.

3. **Main Result**

Definition 9. Let $(A_\zeta, *, 0)$ be a A_ζ. A FS in A_Z with δ_{A_ζ} is said to be **FDzI** of A_ζ if it satisfies

- δ_{A_ζ} $(0, z\zeta) \leq \delta_{A_\zeta}$ $(\omega\zeta, z\zeta)$
- δ_{A_ζ} $(\omega\zeta, z\zeta) \leq \max\{\delta_{A_\zeta}(\omega\zeta, z\zeta), \delta_{A_\zeta}$ $(\kappa\zeta, z\zeta)\}$ for all $\omega\zeta, \kappa\zeta, \eta\zeta \in A_Z$, $p\zeta \in$ p

Definition 10. Let $(A_\zeta,*,0)$ be a A_Z. A FS in A_Z with δ_{A_ζ} is said to be **FDhI** of A_ζ if it satisfies

- δ_{A_ζ} $(0, h\zeta) \leq \delta_{A_\zeta}$ $(\omega\zeta, h\zeta)$
- δ_{A_ζ} $(\omega\zeta, h\zeta) \leq \max\{\delta_{A_\zeta}$ $((\omega\zeta * (\kappa\zeta * \eta\zeta)), h\zeta), \delta_{A_\zeta}$ $(\kappa\zeta, h\zeta)\}$ for all $\omega\zeta, \kappa\zeta, \eta\zeta \in A_Z$, $h\zeta \in$ h

Definition 11. A FS in A_Z with $\delta_{A_\zeta} = (A_\zeta,*\ ,0)$ is said to be **FDpI** of A_ζ if it satisfies

- δ_{A_ζ} $(0, p\zeta) \leq \delta_{A_\zeta}$ $(\omega\zeta, p\zeta)$
- δ_{A_ζ} $(\omega\zeta, p\zeta) \leq \max\{\delta_{A_\zeta}$ $((\omega\zeta * \kappa\zeta) * (\eta\zeta * \kappa\zeta)), h\zeta), \delta_{A_\zeta}$ $(\kappa\zeta, h\zeta)\}$ for all $\omega\zeta, \kappa\zeta, \eta\zeta \in A_Z$, $p\zeta \in$ p.

Theorem 1 Let B_ζ in A_Z satisfies $\omega_\zeta * \kappa_\zeta = (\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta)$ for all $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_Z$. Then Every FDzI of B_ζ is a FDpI of B_ζ.

Proof.

Assume that B_ζ is a FDzI of A_Z.

Such that for all $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_Z$, $p_\zeta \in p$.

$$\delta_{B_\zeta}(0, p_\zeta) \leq \delta_{B_\zeta}(\omega_\zeta, p_\zeta) \quad (1)$$

$$\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \leq \max\left\{\delta_{B_\zeta}(\omega_\zeta * \kappa_\zeta, p_\zeta), \delta_{B_\zeta}(\kappa_\zeta), p_\zeta\right\}$$

$$= \max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_\zeta}(\kappa_\zeta, p_\zeta)\right\}$$

$$\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \leq \max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_\zeta}(\kappa_\zeta, p_\zeta)\right\} \quad (2)$$

From (1) and (2), We get B_ζ is a FDpI of A_Z.

Example 1 Suppose $A_Z = \{0, \omega_\zeta, \kappa_\zeta, \eta_\zeta\}$ the operation given by the Table 1.

TABLE 1

$*$	0	$\omega\zeta$	$\kappa\zeta$	$\eta\zeta$
0	0	$\omega\zeta$	$\kappa\zeta$	$\eta\zeta$
$\omega\zeta$	0	$\omega\zeta$	$\kappa\zeta$	$\omega\zeta$
$\kappa\zeta$	0	$\kappa\zeta$	$\kappa\zeta$	$\omega\zeta$
$\eta\zeta$	0	$\omega\zeta$	$\omega\zeta$	$\eta\zeta$

Hence $(A_z,*,0)$ is a Z -algebra.

We define $F_\zeta : A_Z \to [0,1]$ by

$$\delta_\zeta(0, p_\zeta) = 0.4, \delta_\zeta(\omega_\zeta, p_\zeta) = 0.5, \delta_\zeta(\kappa_\zeta, p_\zeta) = 0.6, \delta_\zeta(\eta_\zeta, p_\zeta) = 0.7$$

Theorem 2 Let A_ζ be a medial A_Z. Then every FDpI of A_Z is a FDzI of A_Z

Example 2 By above table, theorem 1and 2 proved.

Theorem 3 Let B_ζ be a medial A_Z, then B_ζ be a FDpI of A_Z iff B_ζ is a FDhI of A_Z.

Theorem 4 Let A_Z be a FDpI iff for any $\tau \in [0,1]$, $\cup_\zeta\left(\delta_{B_\zeta};\tau\right) = \left\{\omega_\zeta \in A_\zeta, p_\zeta \in p/\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \le \tau\right\}$ is a p-ideal of A_Z. Where $\cup_\zeta\left(\delta_{B_\zeta}:\tau\right) \ne \phi$.

Proof.

Let B_ζ is a FDpI of A_Z and $\cup_\zeta\left(\delta_{B_\zeta}:\tau\right) \ne \phi$ for any $\tau \in [0,1]$.

Let $(\omega_\zeta, p_\zeta) \in \cup_\zeta\left(\delta_{B_\zeta}:\tau\right)$ and $\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \le \tau$

By definition, $\delta_{B_\zeta}(0, p_\zeta) \le \delta_{B_\zeta}(\omega_\zeta, p_\zeta) \le \tau$

Thus $0 \in \cup_\zeta\left(\delta_{A_\zeta}:\tau\right)$

If $\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right) \in \cup_\zeta\left(\delta_{B_\zeta}:\tau\right)$ and $\kappa_\zeta \in \cup_\zeta\left(\delta_{B_\zeta}:\tau\right)$

Then $\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right) \le \tau$ and $\delta_{B_\zeta}(\kappa_\zeta, p_\zeta) \le \tau$.

By definition, $\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \le \max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_\zeta}(\kappa_\zeta, p_\zeta)\right\}$

$$\le \max\{\tau, \tau\} = t$$

Therefore, $\omega_\zeta \in \cup_\zeta\left(\delta_{A_\zeta}:\tau\right)$

Hence, $\cup_\zeta\left(\delta_{A_\zeta}:\tau\right)$ is a p-ideal of A_Z.

Conversely, Assume that for each $\tau \in [0,1]$, $\cup_\zeta\left(\delta_{B_\zeta}:\tau\right)$ is either a p-ideal or empty of A_ζ, for any $\omega_\zeta \in A_\zeta, p_\zeta \in p$,

let $\delta_{B_\zeta}(\omega_\zeta, p_\zeta) = \tau$, Then $\omega_\zeta \in \cup_\zeta\left(\delta_{B_\zeta}:\tau\right)$.

Since $\cup_{A_z}\left(\delta_{B_\zeta}:\tau\right) \ne 0$ is a p-ideal of A_Z.

We have $0 \in \cup_{A_z}\left(\delta_{A_z}:\tau\right)$ and hence $\delta_{A_z}(0) \le \tau = \delta_{A_z}(\omega_\zeta, p_\zeta)$ for all $\omega_\zeta \in A_Z, p_\zeta \in p$.

$0 \in \cup_{A_z}\left(\delta_{B_\zeta}:\tau\right)$ and hence $\delta_{B_\zeta}(0, p_\zeta) \le \tau = \delta_{B_\zeta}(\omega_{\zeta,}, p_\zeta)$ for all $\omega_\zeta \in A_Z, p_\zeta \in p$.

Assume $\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \le \max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_\zeta}(\kappa_\zeta, p_\zeta)\right\}$ for all $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_Z$ is not true.

Then there exist $\omega_{\zeta_0}, \kappa_{\zeta_0}, \eta_{\zeta_0} \in A_Z, p_\zeta \in p$ such that

$$\delta_{B_\zeta}(\omega_{\zeta_0}, p_{\zeta_0}) > \max\left\{\delta_{B_\zeta}\left((\omega_{\zeta_0} * \eta_{\zeta_0}) * (\kappa_{\zeta_0} * \eta_{\zeta_0}), p_{\zeta_0}\right), \delta_{B_\zeta}(\kappa_{\zeta_0}, p_{\zeta_0})\right\}$$

Then, $\delta_{B_\zeta}(\omega_{\zeta_0}, p_{\zeta_0}) > \tau_0 > \max\left\{\delta_{B_\zeta}\left((\omega_{\zeta_0} * \eta_{\zeta_0}) * (\kappa_{\zeta_0} * \eta_{\zeta_0}), p_{\zeta_0}\right), \delta_{B_\zeta}(\kappa_{\zeta_0}, p_{\zeta_0})\right\}$

This implies that $\delta_{B_\zeta}(\omega_{\zeta_0}, p_{\zeta_0}) > \max\left\{\delta_{B_\zeta}\left((\omega_{\zeta_0} * \eta_{\zeta_0}) * (\kappa_{\zeta_0} * \eta_{\zeta_0}), p_{\zeta_0}\right), \delta_{B_\zeta}(\kappa_{\zeta_0}, p_{\zeta_0})\right\}$

This is a contradiction.

Therefore, $\delta_{B_\zeta}(\omega_\zeta, p_{\zeta_0}) \leq \max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_{\zeta_0}\right), \delta_{B_\zeta}(\kappa_\zeta, p_{\zeta_0})\right\}$ for all $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_Z, p_{\zeta_0} \in p$.

Hence, B_ζ is a FDpI of A_Z.

Theorem 5 Let A_ζ is a FDpI iff every nonempty upper level subset $\cup_\zeta\left(\delta_{B_\zeta}; \tau\right)$ of $p_\zeta \in p, A_Z$, $\tau \in Im(A)$ is a p-ideal.

Proof.

Let B_ζ be a FpI ofA_Z.

Assume $\cup_\zeta\left(\delta_{B_\zeta}: \tau\right), \tau \in Im(B_\zeta)$ is a p-ideal.

Since $\cup_\zeta\left(\delta_{B_\zeta}: \tau\right) \neq 0$, there exists $\omega_\zeta \in \cup_\zeta\left(\delta_{B_\zeta}: \tau\right)$ such that $\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \leq \tau$.

Since B_ζ is a FDpI of A_Z, $\delta_{B_\zeta}(0, p_\zeta) \leq \delta_{B_\zeta}(\omega_\zeta, p_\zeta)$ for all $\omega_\zeta \in A_Z, p_\zeta \in p$.

Hence for this $\omega_\zeta \in \left(\delta_{B_\zeta}: \tau\right)$,

$\delta_{B_\zeta}(0, p_\zeta) \leq \tau$, implies that $0, p_\zeta \in \cup_\zeta\left(\delta_{B_\zeta}: \tau\right)$.

Now, for any $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_Z$, $p_\zeta \in p$

Assume that $\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right) \in \cup_\zeta\left(\delta_{B_\zeta}: \tau\right)$ and $\kappa_\zeta \in \cup_\zeta\left(\delta_{B_\zeta}: \tau\right)$, $p_\zeta \in p$.

Then $\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right) \leq \tau$ and $\delta_{B_\zeta}(\kappa_\zeta, p_\zeta) \leq \tau$.

Therefore $\max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_\zeta}(\kappa_\zeta, p_\zeta)\right\} \leq \tau$.

Since B_ζ is a FDpI of A_ζ.

$$\delta_{B_\zeta}(\omega_\zeta, p_\zeta) \leq \max\left\{\delta_{B_\zeta}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_\zeta}(\kappa_\zeta, p_\zeta)\right\} \leq \tau.$$

Thus, $\omega_\zeta, p_\zeta \in \left(\delta_{B_\zeta}: \tau\right)$.

Hence, $\cup_\zeta\left(\delta_{B_\zeta}: \tau\right)$ is a p-ideal of A_Z.

Conversely, Let $\cup_\zeta\left(\delta_{B_\zeta}: \tau\right), \tau \in Im(B_\zeta), p_\zeta \in p$ be a p-ideal of A_Z.

Assume B_ζ is a FDpI of A_Z.

Let $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_Z$, $p_\zeta \in p$ for any $\tau \in Im(B_\zeta)$.

Let $\tau = \max\left\{\delta_{B_\zeta}\left(\left(\omega_\zeta * \eta_\zeta\right) * \left(\kappa_\zeta * \eta_\zeta\right), p_\zeta\right), \delta_{B_\zeta}\left(\kappa_\zeta, p_\zeta\right)\right\}$

Therefore, $\delta_{B_\zeta}\left(\left(\omega_\zeta * \eta_\zeta\right) * \left(\kappa_\zeta * \eta_\zeta\right), p_\zeta\right) \leq \tau$ and $\delta_{B_\zeta}\left(\kappa_\zeta, p_\zeta\right) \leq \tau$.

Implies that $\left(\left(\omega_\zeta * \eta_\zeta\right) * \left(\kappa_\zeta * \eta_\zeta\right), p_\zeta\right), \left(\kappa_\zeta, p_\zeta\right) \in \cup_\zeta \left(\delta_{B_\zeta} : \tau\right)$

Since $\cup_\zeta \left(\delta_{B_\zeta} : \tau\right)$ is a p-ideal.

We have $\omega_\zeta : \cup_\zeta \left(\delta_{B_\zeta} : \tau\right)$.

This proves that $\delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) \leq \tau \leq \max\left\{\delta_{B_\zeta}\left(\left(\omega_\zeta * \eta_\zeta\right) * \left(\kappa_\zeta * \eta_\zeta\right), p_\zeta\right), \delta_{B_\zeta}\left(\kappa_\zeta, p_\zeta\right)\right\}$.

Hence, B_ζ is a FDpI of A_Z.

Theorem 6 Let B_ζ is a FDpI of A_Z and let $\omega_\zeta \in A_Z, p_\zeta \in p$. Then $\delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) = \tau$, iff $\left(\omega_\zeta, p_\zeta\right) \in \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ but $\left(\omega_\zeta, p_\zeta\right) \notin \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ for all $\tau_1 < \tau$.

Proof.

Assume $\delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) = \tau$. So that $\left(\omega_\zeta, p_\zeta\right) \in \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$.

If possible let $\left(\omega_\zeta, p_\zeta\right) \in \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ for $\tau_1 \leq \tau$. Then $\delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) \leq \tau_1 < \tau$.

Which is a contradiction that $\delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) = \tau$.

Hence, $\left(\omega_\zeta, p_\zeta\right) \notin \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ for all $\tau_1 < \tau$.

Conversely, Let $\left(\omega_\zeta, p_\zeta\right) \in \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ but $\left(\omega_\zeta, p_\zeta\right) \notin \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ for all $\tau_1 < \tau$.

$$\left(\omega_\zeta, p_\zeta\right) \in \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right) \Longrightarrow \delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) \leq \tau.$$

Since $\left(\omega_\zeta, p_\zeta\right) \notin \cup_\zeta \left(\delta_{B_\zeta} ; \tau\right)$ for all $\tau_1 < \tau$, $\delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right) = \tau$.

Theorem 7 Let $h_\zeta : \left(A_\zeta, *, 0\right) \to \left(B_\zeta, *', 0\right)$ be a Z-homomorphism of A_Z. If B_{ζ_1} is a FDpI of B_ζ. Then $h_\zeta\left(B_{\zeta_1}\right)$ is a FDpI of A_ζ.

Proof.

Since B_{ζ_1} is a FpI of B_ζ.

For any $\omega_\zeta \in A_\zeta, p_\zeta \in p$ we have

$$\begin{aligned} \delta_{h_\zeta^{-1}(B_{\zeta_1})}\left(\omega_\zeta, p_\zeta\right) &= \delta_{B_{\zeta_1}}\left(h_\zeta\left(\omega_\zeta, p_\zeta\right)\right) \\ &\geq \delta_{B_{\zeta_1}}\left(0', p_\zeta\right) \\ &= \delta_{B_{\zeta_1}}\left(h_\zeta\left(0, p_\zeta\right)\right) \\ &= \delta_{h_\zeta^{-1}(B_{\zeta_1})}\left(0, p_\zeta\right) \end{aligned}$$

Let $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_\zeta$,

Then, $\max\left\{\delta_{h_\zeta^{-1}(B_{\zeta_1})}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{h_\zeta^{-1}(B_{\zeta_1})}(\kappa_\zeta, p_\zeta)\right\}$

$$= \max\left\{\delta_{B_{\zeta_1}} h_\zeta\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{B_{\zeta_1}} h_\zeta(\kappa_\zeta, p_\zeta)\right\}$$

$$= \max\left\{\delta_{B_{\zeta_1}}\left(\left(h_\zeta(\omega_\zeta) *' h_\zeta(\eta_\zeta)\right) * \left(h_\zeta(\kappa_\zeta) *' h_\zeta(\eta_\zeta)\right), p_\zeta\right), \delta_{B_{\zeta_1}} h_\zeta(\kappa_\zeta, p_\zeta)\right\}$$

$$= \delta_{B_{\zeta_1}}\left(h_\zeta(\omega_\zeta, p_\zeta)\right)$$

$$= \delta_{h_\zeta^{-1}}\left(B_{\zeta_1}(\omega_\zeta, p_\zeta)\right)$$

From (i) and (ii) we get $h_\zeta^{-1}(B_{\zeta_1})$ is a FDpI.

Theorem 8 Consider $h_\zeta: (A_\zeta, *, 0) \to (B_\zeta, *', 0')$ be a Z-epimorphism of A_Z. Let B_{ζ_1} be a FS of B_ζ. If $h_\zeta^{-1}(B_{\zeta_1})$ is a FDpI of A_ζ.

Proof.

Since B_{ζ_1} is a FDpI of B_ζ.then $h_\zeta^{-1}(B_{\zeta_1})$=$h_\zeta(\omega_\zeta, p_\zeta)$

For any $\omega_\zeta \in A_\zeta, p_\zeta \in p$ we have, $\delta_{h_\zeta^{-1}(B_{\zeta_1})} = \delta_{B_{\zeta_1}}\left(h_\zeta(\omega_\zeta, p_\zeta)\right)$

$$\geq \delta_{B_{\zeta_1}}(0', p_\zeta)$$

$$= \delta_{B_{\zeta_1}}\left(h_\zeta(0, p_\zeta)\right)$$

$$= \delta_{h_\zeta^{-1}(B_{\zeta_1})}(0, p_\zeta)$$

Let $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in A_\zeta, p_\zeta \in p$ then then there exist $\alpha_\zeta, \beta_\zeta, \gamma_\zeta \in B_\zeta$.

such that $h_\zeta(\alpha_\zeta, p_\zeta) = (\omega_\zeta, p_\zeta)$, $h_\zeta(\beta_\zeta, p_\zeta) = (\kappa_\zeta, p_\zeta), h_\zeta(\gamma_\zeta, p_\zeta) = (\eta_\zeta, p_\zeta)$

Now, $\delta_{B_{\zeta_1}}(\omega_\zeta, p_\zeta) = \delta_{B_{\zeta_1}}\left(h_\zeta(\alpha_\zeta, p_\zeta)\right)$

$$= \max\left\{\delta_{h_\zeta^{-1}(B_{\zeta_1})}\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta), p_\zeta\right), \delta_{h_\zeta^{-1}(B_{\zeta_1})}(\kappa_\zeta, p_\zeta)\right\}$$

$$= \max\left\{\delta_{B_{\zeta_1}}\left(h_\zeta\left((\omega_\zeta * \eta_\zeta) * (\kappa_\zeta * \eta_\zeta)\right), p_\zeta\right), \delta_{B_{\zeta_1}}\left(h_\zeta(\kappa_\zeta, p_\zeta)\right)\right\}$$

$$= \max\left\{\delta_{B_{\zeta_1}}\left(\left(h_\zeta(\omega_\zeta) *' h_\zeta(\eta_\zeta)\right) * \left(h_\zeta(\kappa_\zeta) *' h_\zeta(\eta_\zeta)\right), p_\zeta\right), \delta_{B_{\zeta_1}} h_\zeta(\kappa_\zeta, p_\zeta)\right\}$$

$$= \delta_{B_{\zeta_1}}\left(h_\zeta(\omega_\zeta, p_\zeta)\right)$$

$$= \delta_{h_\zeta^{-1}(B_{\zeta_1})}(\omega_\zeta, p_\zeta)$$

Hence, $h_\zeta^{-1}(B_{\zeta_1})$ is a FDpI of A_ζ.

Theorem 9 Let $h_\zeta: (A_\zeta, *, 0) \to (B_\zeta, *', 0')$ be a Z-epimorphism of A_ζ. Let B_{ζ_1} be a fuzzy set of B_ζ. If $h_\zeta^{-1}(B_{\zeta_1})$ is a FDpI of A_ζ, then B_{ζ_1} is a FDpI of B_ζ.

Proof.

Let $\kappa_\zeta \in B_\zeta \ \in A_\zeta, p_\zeta \in p$ such that $h_\zeta(\omega_\zeta, p_\zeta) = (\omega_\zeta, p_\zeta)$. Then

$$\begin{aligned}\delta_{B_{\zeta_1}}(\kappa_\zeta, p_\zeta) &= \delta_{B_{\zeta_1}}\left(h_\zeta(\omega_\zeta, p_\zeta)\right)\\ &= \delta_{h_\zeta^{-1}(B_{\zeta_1})}(\omega_{\zeta,}, p_\zeta)\\ &\geq \delta_{h_\zeta^{-1}(B_{\zeta_1})}(0, p_\zeta)\\ &= \delta_{B_{\zeta_1}} h_\zeta(0, p_\zeta)\\ &= \delta_{B_{\zeta_1}} h_\zeta(0', p_\zeta)\end{aligned}$$

This implies $\delta_{B_{\zeta_1}} h_\zeta(0', p_\zeta) \leq \delta_{B_{\zeta_1}}(\kappa_\zeta, p_\zeta)$.

Let $\omega_\zeta, \kappa_\zeta, \eta_\zeta \in B_\zeta, p_\zeta \in p$ then there exist $\alpha_\zeta, \beta_\zeta, \gamma_\zeta \in A_\zeta$,

Such that $h_\zeta(\alpha_\zeta, p_\zeta) = (\omega_\zeta, p_\zeta), h_\zeta(\beta_\zeta, p_\zeta) = (\kappa_\zeta, p_\zeta), h_\zeta(\gamma_\zeta, p_\zeta) = (\eta_\zeta, p_\zeta)$.

Now, $\delta_{B_{\zeta_1}}(\omega_\zeta, p_\zeta) = \delta_{B_{\zeta_1}}\left(h_\zeta(\alpha_\zeta)\right)$

$$\begin{aligned}&\leq \max\left\{\delta_{h_\zeta^{-1}(B_{\zeta_1})}\left((\alpha_\zeta * \gamma_\zeta) * (\beta_\zeta * \gamma_\zeta), p_\zeta\right), \delta_{h_\zeta^{-1}(B_{\zeta_1})}(\beta_{\zeta,}, p_\zeta)\right\}\\ &= \max\left\{\delta_{B_{\zeta_1}}\left(h_\zeta\left((\alpha_\zeta * \gamma_\zeta) * (\beta_\zeta * \gamma_\zeta)\right), p_\zeta\right), \delta_{B_{\zeta_1}}\left(h_\zeta(\beta_{\zeta,}, p_\zeta)\right)\right\}\\ &= \max\left\{\delta_{B_{\zeta_1}}\left(\left(h_\zeta(\alpha_\zeta) *' h_\zeta(\gamma_\zeta)\right) * \left(h_\zeta(\beta_\zeta) *' h_\zeta(\gamma_\zeta)\right), p_\zeta\right), \delta_{B_{\zeta_1}}\left(h_\zeta(\beta_\zeta, p_\zeta)\right)\right\}\\ &= \max\left\{\delta_{B_{\zeta_1}}\left((\omega_\zeta *' \eta_\zeta) * (\kappa_\zeta *' \eta_\zeta), p_\zeta\right), \delta_{B_{\zeta_1}}\left(h_\zeta(\kappa_\zeta), p_\zeta\right)\right\}\end{aligned}$$

Which implies that

$$\delta_{B_{\zeta_1}}(\omega_\zeta, p_\zeta) \leq \max\left\{\delta_{B_{\zeta_1}}\left((\omega_\zeta *' \eta_\zeta) * (\kappa_\zeta *' \eta_\zeta), p_\zeta\right), \delta_{B_{\zeta_1}}\left(h_\zeta(\kappa_\zeta), p_\zeta\right)\right\}$$

Hence, B_{ζ_1} is a FDpI of B_ζ.

Theorem 10 Let h_ζ be an Z- epimorphism of A_Z and let A_ζ be a FS in A_Z. Then $A_\zeta^{h_\zeta}: A_Z \to [0,1]$ defined by $\delta_{A_\zeta}^{h_\zeta}(\omega_\zeta) = \delta_{A_\zeta}^{h_\zeta}\left(h_\zeta(\omega_\zeta)\right)$ for all $\omega_\zeta \in A_Z$ is a FDpI of A_Z if $A_\zeta$is a FDpI.

Theorem 11 Let A_ζ and B_ζ be a FDpI of A_Z then $A_\zeta \times B_\zeta$ is a FDpI in $A_Z \times A_Z$.

Proof.

Let $\left(\omega_{\zeta_1}, \omega_{\zeta_2}\right) \in (A_Z \times A_Z)$

$$\begin{aligned}\delta_{A_\zeta \times B_\zeta}(0,0) &= \max\left\{\delta_{A_\zeta}(0, p_{\zeta_1}), \delta_{B_\zeta}(0, p_{\zeta_2})\right\}\\ &\leq \max\left\{\delta_{A_\zeta}(\omega_{\zeta_1}, p_{\zeta_1}), \delta_{B_\zeta}\left(\omega_{\zeta_2}, p_{\zeta_2}\right)\right\}\\ &= \delta_{A_\zeta \times B_\zeta}\left((\omega_{\zeta_1}, p_{\zeta_1}), \left(\omega_{\zeta_2}, p_{\zeta_2}\right)\right)\end{aligned}$$

Hence $\delta_{A_\zeta \times B_\zeta}(0,0) \leq \delta_{A_\zeta \times B_\zeta}\left(\left(\omega_{\zeta_1}, p_{\zeta_1}\right), \left(\omega_{\zeta_2}, p_{\zeta_2}\right)\right)$.

Let $\left(\omega_{\zeta_1}, \omega_{\zeta_2}\right), \left(\kappa_{\zeta_1}, \kappa_{\zeta_2}\right), \left(\eta_{\zeta_1}, \eta_{\zeta_2}\right) \in A_Z \times A_Z$, $p_{\zeta_2} \in p$

Then, $\delta_{A_\zeta \times B_\zeta}\left(\left(\omega_{\zeta_1}, p_{\zeta_1}\right), \left(\omega_{\zeta_2}, p_{\zeta_2}\right)\right) = \max\left\{\delta_{A_\zeta}\left(\omega_{\zeta_1}, p_{\zeta_1}\right), \delta_{B_\zeta}\left(\omega_{\zeta_2}, p_{\zeta_2}\right)\right\}$

$$\leq \max\left\{\begin{matrix}\max\left\{\delta_{A_\zeta}\left(\left(\omega_{\zeta_1} * \eta_{\zeta_1}\right) * \left(\kappa_{\zeta_1} * \eta_{\zeta_1}\right), p_{\zeta_1}\right), \delta_{A_\zeta}\left(\kappa_{\zeta_1}, p_{\zeta_1}\right)\right\}, \\ \max\left\{\delta_{B_\zeta}\left(\left(\omega_{\zeta_2} * \eta_{\zeta_2}\right) * \left(\kappa_{\zeta_2} * \eta_{\zeta_2}\right), p_{\zeta_2}\right), \delta_{B_\zeta}\left(\kappa_2, p_{\zeta_2}\right)\right\}\end{matrix}\right\}$$

$$= \max\left\{\begin{matrix}\max\left\{\begin{matrix}\delta_{A_\zeta}\left(\left(\omega_{\zeta_1} * \eta_{\zeta_1}\right) * \left(\kappa_{\zeta_1} * \eta_{\zeta_1}\right), p_{\zeta_1}\right), \\ \delta_{B_\zeta}\left(\left(\omega_{\zeta_2} * \eta_{\zeta_2}\right) * \left(\kappa_{\zeta_2} * \eta_{\zeta_2}\right) p_{\zeta_2}\right)\end{matrix}\right\}, \\ \max\left\{\delta_{A_\zeta}\left(\kappa_{\zeta_1}, p_{\zeta_1}\right), \delta_{B_\zeta}\left(\kappa_{\zeta_2}, p_{\zeta_2}\right)\right\}\end{matrix}\right\}$$

$$= \max\left\{\begin{matrix}\delta_{A_\zeta \times B_\zeta}\left(\begin{matrix}\left(\left(\omega_{\zeta_1} * \eta_{\zeta_1}\right) * \left(\kappa_{\zeta_1} * \eta_{\zeta_1}\right), p_{\zeta_1}\right), \\ \left(\left(\omega_{\zeta_2} * \eta_{\zeta_2}\right) * \left(\kappa_{\zeta_2} * \eta_{\zeta_2}\right)\right), p_{\zeta_2}\end{matrix}\right), \\ \delta_{A_\zeta \times B_\zeta}\left(\kappa_{\zeta_1}, p_{\zeta_1}, \kappa_{\zeta_2}, p_{\zeta_2}\right)\end{matrix}\right\}$$

$$= \max\left\{\begin{matrix}\delta_{A_\zeta \times B_\zeta}\left(\begin{matrix}\left(\left(\omega_{\zeta_1}, \omega_{\zeta_2}\right) * \left(\eta_{\zeta_1}, \eta_{\zeta_2}\right)\right) * \\ \left(\left(\kappa_{\zeta_1}, \kappa_{\zeta_2}\right) * \left(\eta_{\zeta_1}, \eta_{\zeta_2}\right)\right), p_{\zeta_1}, p_{\zeta_2}\end{matrix}\right), \\ \delta_{A_\zeta \times B_\zeta}\left(\kappa_{\zeta_1}, p_{\zeta_1}, \kappa_{\zeta_2}, p_{\zeta_2}\right)\end{matrix}\right\}$$

Hence $A_\zeta \times B_\zeta$ is a FDpI in $A_Z \times A_Z$.

Theorem 12 Let A_ζ and B_ζ is a FS in A_Z such that $A_\zeta \times B_\zeta$ is a FDpI of $A_Z \times A_Z$. Then

(i) Either $\delta_{A_\zeta}\left(0, p_\zeta\right) \leq \delta_{A_\zeta}\left(\omega_\zeta, p_\zeta\right)$ or $\delta_{B_\zeta}\left(0, p_\zeta\right) \leq \delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right)$ for all $\omega_\zeta, p_\zeta \in A_Z$.

(ii) If $\delta_{A_\zeta}\left(0, p_\zeta\right) \leq \delta_{A_\zeta}\left(\omega_\zeta, p_\zeta\right)$ for all $\omega_\zeta \in A_Z$. Then either $\delta_{A_\zeta}\left(0, p_\zeta\right) \leq \delta_{A_\zeta}\left(\omega_\zeta, p_\zeta\right)$ or $\delta_{A_\zeta}\left(0, p_\zeta\right) \leq \delta_{A_\zeta}\left(\omega_\zeta, p_\zeta\right)$.

(iii) If $\delta_{B_\zeta}\left(0, p_\zeta\right) \leq \delta_{B_\zeta}\left(\omega_\zeta, p_\zeta\right)$ for all $\omega_\zeta \in A_Z$. Then either $\delta_{A_\zeta}\left(0, p_\zeta\right) \leq \delta_{A_\zeta}\left(\omega_\zeta, p_\zeta\right)$ or $\delta_{A_\zeta}\left(0, p_\zeta\right) \leq \delta_{A_\zeta}\left(\omega_\zeta, p_\zeta\right)$.

Proof.

(i) $\delta_{A_\zeta}\left(0, p_{\zeta_1}\right) > \delta_{A_\zeta}\left(\omega_{\zeta_1}, p_{\zeta_1}\right)$ and $\delta_{B_\zeta}(0) \leq \delta_{B_\zeta}\left(\omega_{\zeta_2}\right)$ for some $\omega_{\zeta_1}, \omega_{\zeta_2} \in A_Z$.

Then,

$$\delta_{A_\zeta \times B_\zeta}\left(\omega_{\zeta_1}, p_{\zeta_1}, \omega_{\zeta_2}\right) = \max\left\{\delta_{A_\zeta}\left(\omega_{\zeta_1}, p_{\zeta_1}\right), \delta_{B_\zeta}\left(\omega_{\zeta_2}, p_{\zeta_2}\right)\right\}$$
$$< \max\left\{\delta_{A_\zeta}\left(0, p_{\zeta_1}\right), \delta_{B_\zeta}\left(0, p_{\zeta_2}\right)\right\}$$
$$= \delta_{A_\zeta \times B_\zeta}\left(0, p_{p_{\zeta_2}}\right)$$

Which is a contradiction.

Hence, Either $\delta_{A_\zeta}(0, p_\zeta) \leq \delta_{A_\zeta}(\omega_\zeta, p_\zeta)$ or $\delta_{B_\zeta}(0) \leq \delta_{B_\zeta}(\omega_\zeta, p_\zeta)$ for all $\omega_\zeta \in A_Z, p_\zeta \in p$.

(ii) Let $\delta_{A_\zeta}(0,, p_{\zeta_1}) \leq \delta_{A_\zeta}(\omega_{\zeta_1}, p_{\zeta_1})$ for all $\omega_\zeta \in A_Z, p_\zeta \in p$. Assume that there exist $\omega_{\zeta_1}, \omega_{\zeta_2} \in A_Z, p_{\zeta_1}, p_{\zeta_2} \in p_Z$ such that $\delta_{A_\zeta}(0, p_{\zeta_1}) > \delta_{B_\zeta}(\omega_\zeta)$ or $\delta_{B_\zeta}(0) > \delta_{A_\zeta}(\omega_{\zeta_1}, p_{\zeta_1})$. Then

$$\delta_{A_\zeta \times B_\zeta}(0_\zeta, p_\zeta) = \max\left\{\delta_{A_\zeta}(0, p_{\zeta_1}), \delta_{B_\zeta}(0)\right\}$$
$$= \delta_{B_\zeta}(0)$$
$$\delta_{A_\zeta \times B_\zeta}(\omega_\zeta, p_\zeta) = \max\left\{\delta_{A_\zeta}(\omega_{\zeta_1}, p_{\zeta_1}), \delta_{B_\zeta}(\omega_{\zeta_2}, p_{\zeta_2})\right\}$$
$$< \delta_{B_\zeta}(0, p_{\zeta_2})$$
$$= \delta_{A_\zeta \times B_\zeta}(0_\zeta, p_\zeta)$$
$$\delta_{A_\zeta \times B_\zeta}(\omega_\zeta, p_\zeta) < \delta_{A_\zeta \times B_\zeta}(0, p_\zeta)$$

Which is a contradiction.

Hence proved for all $\omega_\zeta, p_{\zeta_1} \in A_Z$.

(iii) Similarly, proved by (ii)

Theorem 13 Let A_ζ and B_ζ be a FS in $A_Z, A_\zeta * B_\zeta$ be a FDpI of $A_Z \times A_Z$. Then either A_ζ or B_ζ is a FDpI of A_Z.

Proof.

By above theorem,

We assume $\delta_{B_\zeta}(0, p_{\zeta_2}) \leq \delta_{B_\zeta}(\omega_\zeta, p_{\zeta_2})$ for all $\omega_\zeta \in A_Z$. Then either $\delta_{A_\zeta}(0, p_\zeta) \leq \delta_{A_\zeta}(\omega_\zeta, p_\zeta)$ or $\delta_{B_\zeta}(0, p_{\zeta_2}) \leq \delta_{B_\zeta}(\omega_\zeta, p_{\zeta_2})$.

Let $\delta_{A_\zeta}(0, p_\zeta) \leq \delta_{B_\zeta}(\omega_{\zeta_2}, p_{\zeta_2})$ for any $\omega_\zeta \in A_Z$.

Then, $\delta_{B_\zeta}(\omega_{\zeta_2}, p_{\zeta_2}) = \max\left\{\delta_{B_\zeta}(0, p_{\zeta_2}), \delta_{B_\zeta}(\omega_{\zeta_2}, p_{\zeta_2})\right\}$

$$= \delta_{A_\zeta \times B_\zeta}(0, p_\zeta, \omega_\zeta, p_\zeta)$$
$$\leq \max\left\{\begin{matrix}\delta_{A_\zeta \times B_\zeta}\left(\left((0, \omega_\zeta) * (0, \eta_\zeta) p_\zeta\right) * \left((0, p_\zeta, \kappa_\zeta, p_\zeta) * (0, p_\zeta, \eta_{\zeta,}, p_\zeta)\right)\right), \\ \delta_{A_\zeta \times B_\zeta}(0, p_\zeta \kappa_\zeta, p_\zeta)\end{matrix}\right\}$$
$$= \max\left\{\begin{matrix}\delta_{A_\zeta \times B_\zeta}\left(\left((0, p_\zeta * 0, p_\zeta), (\omega_\zeta, p_\zeta * \eta_\zeta, p_\zeta)\right) * \left((0, p_\zeta * 0, p_\zeta), (\kappa_{\zeta,}, p_\zeta * \eta_\zeta, p_\zeta)\right)\right), \\ \delta_{A_\zeta \times B_\zeta}(0, p_\zeta, \kappa_\zeta, p_\zeta)\end{matrix}\right\}$$

$$= \max\left\{\delta_{A_\zeta\times B_\zeta}\left((0*0,p_\zeta),\left((\omega_\zeta*\eta_\zeta)*(\kappa_\zeta*\eta_\zeta)\right),p_\zeta\right),\delta_{A_\zeta\times B_\zeta}(0,p_\zeta,\kappa_\zeta,p_\zeta)\right\}$$

$$= \max\left\{\begin{matrix}\max\left\{\delta_{A_\zeta}(0*0,p_\zeta),\delta_{B_\zeta}\left((\omega_\zeta*\eta_\zeta)*(\kappa_\zeta*\eta_\zeta),p_\zeta\right)\right\},\\ \min\left\{\delta_{A_\zeta}(0,p_\zeta),\delta_{B_\zeta}(\kappa_\zeta,p_\zeta)\right\}\end{matrix}\right\}$$

$$= \max\left\{\delta_{B_\zeta}\left((\omega_\zeta*\eta_\zeta)*(\kappa_\zeta*\eta_\zeta),p_\zeta\right),\delta_{B_\zeta}(\kappa_\zeta,p_\zeta)\right\}$$

Therefore,

$\delta_{B_\zeta}(\omega_\zeta) \le \max\left\{\delta_{B_\zeta}\left((\omega_\zeta*\eta_\zeta)*(\kappa_\zeta*\eta_\zeta),p_\zeta\right),\delta_{B_\zeta}(\kappa_\zeta,p_\zeta)\right\}$, for all $\omega_\zeta,\kappa_\zeta,\eta_\zeta \in A_Z$.

Hence, B_ζ is a FDpI of A_Z.

By above theorem, Assume that $\delta_{A_\zeta}(0,p_\zeta) \le \delta_{A_\zeta}(\omega_\zeta,p_\zeta)$ for all $\omega_\zeta,p_\zeta \in A_Z$ and $\delta_{B_\zeta}(0,p_\zeta) \le \delta_{B_\zeta}(\omega_\zeta,p_\zeta)$ for all $\omega_\zeta \in A_Z$.

Then, A_ζ is a FDpI of A_Z.

Hence, proved.

Theorem 14. Let A_{ζ_1} and A_{ζ_2} be two FDpI of A_Z, then $A_{\zeta_1} \cap A_{\zeta_2}$ is a FDpI of A_Z.

Example 2 Suppose A_Z= $\{0,\omega_\zeta,\kappa_\zeta,\eta_\zeta\}$ the operation given by the Table 1.

A_{ζ_1} *	0	$\omega\zeta$	$\kappa\zeta$	$\eta\zeta$	A_{ζ_2} *	0	$\omega\zeta$	$\kappa\zeta$	$\eta\zeta$
0	0	$\omega\zeta$	$\kappa\zeta$	$\eta\zeta$	**0**	0	$\omega\zeta$	$\kappa\zeta$	$\eta\zeta$
$\omega\zeta$	0	$\omega\zeta$	$\kappa\zeta$	$\omega\zeta$	$\omega\zeta$	0	$\omega\zeta$	$\kappa\zeta$	$\omega\zeta$
$\kappa\zeta$	0	$\kappa\zeta$	$\kappa\zeta$	$\eta\zeta$	$\kappa\zeta$	0	$\kappa\zeta$	$\kappa\zeta$	$\eta\zeta$
$\eta\zeta$	0	$\omega\zeta$	$\eta\zeta$	$\eta\zeta$	$\eta\zeta$	0	$\omega\zeta$	$\eta\zeta$	$\eta\zeta$

Hence $(A_z,*,0)$ is a Z -algebra.

We define $F_\zeta: A_Z \to [0,1]$ by

$$\delta_{\zeta_1}(0,p\zeta) = 0.32,\ \delta_{\zeta_1}(\omega\zeta,p\zeta) = 0.43,\ \delta_{\zeta_1}(\kappa\zeta,p\zeta) = 0.76,\ \delta_{\zeta_1}(\eta\zeta,p\zeta) = 0.87$$

$$\delta_{\zeta_2}(0,p\zeta) = 0.21,\ \delta_{\zeta_2}(\omega\zeta,p\zeta) = 0.54,\ \delta_{\zeta_2}(\kappa\zeta,p\zeta) = 0.65,\ \delta_{\zeta_2}(\eta\zeta,p\zeta) = 0.98$$

$$\delta_{\zeta_1\cup\zeta_2}(0,p\zeta) = 0.32,\ \delta_{\zeta_1\cup\zeta_2}(\omega\zeta,p\zeta) = 0.54,\ \delta_{\zeta_1\cup\zeta_2}(\kappa\zeta,p\zeta) = 0.76,\ \delta_{\zeta_1\cup\zeta_2}(\eta\zeta,p\zeta) = 0.98$$

$$\delta_{\zeta_1\cap\zeta_2}(0, p\,\zeta) = 0.21, \delta_{\zeta_1\cap\zeta_2}(\omega\,\zeta, p\,\zeta) = 0.43, \delta_{\zeta_1\cap\zeta_2}(\kappa\,\zeta, p\,\zeta) = 0.65, \delta_{\zeta_1\cap\zeta_2}(\eta\,\zeta, p\,\zeta)=87$$

Theorem 15. Let A_{ζ_1} and A_{ζ_2} be two FDpI of A_Z, then $A_{\zeta_1} \cup A_{\zeta_2}$ is a FDpI of A_Z.

Corollary 16. Let $A_{\zeta_1}, A_{\zeta_2}, \ldots, A_{\zeta_n}$ be a FDpI of A_ζ, then $A\,\zeta$ is a FDpI of A_Z.

Theorem 17. Let A_{ζ_1} be a FDpI of A_Z, then ${A_{\zeta_1}}^i$ be a FDpI of A_Z, where i denoted positive integer.

CONCLUSION

Through this work, we present the definitions of the Fuzzy Doubt p-ideals and study some relationship among these types. Also, we can study the notion of the Fuzzy Doubt p-Ideals in Z-algebra. We look to extend the idea of Zalgebras in the fuzzy ideal which plays an important role. Also, we proposed several theorem of Fuzzy Doubt p-Ideals in Z-algebras. Along the above notion we can be proceeding for further developments of intuitionistic doubt fuzzy z-algebras in alternative branches of algebra.

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